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Asymptotics for the spectrum of the Wentzell problem with a small parameter and other related stiff problems ☆

D. Gómez [a], M. Lobo [a], S.A. Nazarov [b], E. Pérez [c,*]

[a] *Departamento de Matemáticas, Estadística y Computación, Universidad de Cantabria, Avenida de los Castros s/n, 39005 Santander, Spain*
[b] *Institute of Mechanical Engineering Problems, RAN V.O. Bol'shoi pr. 61, 199178 St. Petersburg, Russia*
[c] *Departamento de Matemática Aplicada y Ciencias de la Computación, Universidad de Cantabria, Avenida de los Castros s/n, 39005 Santander, Spain*

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Abstract

We consider different stiff spectral problems with a small parameter for the Laplace operator in two different domains of the plane Ω and Ω_ε, respectively. Here $\Omega_\varepsilon = \Omega \cup \omega_\varepsilon \cup \Gamma$, where Ω is a fixed open bounded domain with boundary Γ, ω_ε is a curvilinear strip of variable width $O(\varepsilon)$, and $\Gamma = \overline{\Omega} \cap \bar{\omega}_\varepsilon$. ε and δ_ε are small parameters that converge towards zero. The first problem is a *Wentzell spectral problem* in the fixed domain Ω, with the parameter δ_ε appearing on the boundary condition, multiplying the normal derivative on Γ. For the second problem, posed in Ω_ε with a Neumann condition on the boundary of Ω_ε, the *density* and *stiffness constants* are of order $O(\varepsilon^{-t})$ in the strip ω_ε, with $t > 1$, while they are of order $O(1)$ in the fixed domain Ω. We provide asymptotic expansions for the eigenvalues and eigenfunctions of both problems and obtain bounds for convergence rates of these eigenelements as $\varepsilon \to 0$. In addition, we seek out the connection between both problems, which have a common limiting eigenvalue problem (cf. (2.15)–(2.16)), and notice an asymptotic dissociation in two spectral problems on Ω and Γ. We also show that the Wentzell spectral problem can be considered as an alternative approach for the stiff problem in the perturbed domain Ω_ε when $\delta_\varepsilon = \varepsilon^{t-1}$, as $\varepsilon \to 0$.

Résumé

Sur le comportement asymptotique des éléments propres du problème de Wentzell dépendant d'un petit paramètre ainsi que d'autres problèmes spectraux raides. On considère des problèmes spectraux du type raide, avec un petit paramètre (ε or δ_ε), pour l'opérateur de Laplace posés dans deux domaines différents du plan Ω and Ω_ε, respectivement. Ici $\Omega_\varepsilon = \Omega \cup \omega_\varepsilon \cup \Gamma$, où Ω est un domaine borné indépendent du paramètre ε, et à frontière régulière Γ, ω_ε est une couche curviligne autour de Γ, de largeur variable $O(\varepsilon)$, et $\Gamma = \overline{\Omega} \cap \bar{\omega}_\varepsilon$. ε et δ_ε sont des petits paramètres que nous ferons converger vers zéro. Le premier problème est un *problème spectral de Wentzell*, dans le domaine Ω, où le petit paramètre δ_ε apparaît dans la condition aux limites, en multipliant la derivée normale sur Γ. Le second problème, posé dans Ω_ε, est un problème de Neumann, où les constantes relatives à la *densité* et la *raideur* sont d'ordre $O(\varepsilon^{-t})$ dans la bande ω_ε, avec $t > 1$, tandis qu'elles sont d'ordre $O(1)$ dans le domaine Ω. On construit des développements asymptotiques des valeurs propres et des fonctions propres pour les deux problèmes et on obtient des estimations

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* Corresponding author.
E-mail addresses: gomezdel@unican.es (D. Gómez), miguel.lobo@unican.es (M. Lobo), serna@ipme.ru (S.A. Nazarov), meperez@unican.es (E. Pérez).

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pour la vitesse de convergence des éléments propres, lorsque $\varepsilon \to 0$. Par ailleurs, on établit la connexion entre les deux problèmes spectraux : Ils ont le même problème spectral limite très particulier (voir (2.15)–(2.16)), puis qu'ils présentent une dissociation en deux problèmes spectraux dans Ω et dans Γ. De plus, nous montrons que le problème spectral de Wentzell peut être consideré comme une approximation asymptotique alternative, lorsque $\varepsilon \to 0$, du problème raide posé dans le domaine perturbé Ω_ε pour la relation $\delta_\varepsilon = \varepsilon^{t-1}$.

Keywords: Stiff problems; Asymptotic analysis; Spectral analysis; Correcting terms

1. Introduction and statement of the problem

In this paper, we study the asymptotic behavior of the eigenvalues and eigenfunctions of two stiff problems. The first one is a spectral stiff problem in a fixed domain of the plane, the so-called *Wentzell problem*, with the perturbation parameter $\delta_\varepsilon \to 0$ accompanying the normal derivative on the boundary; namely, problem (1.1). The second problem is *a spectral stiff problem in a domain surrounded by a thin band*, the thickness of this band depending on the small parameter ε, while the coefficients are very large in this region; namely, problem (1.2) for $t > 1$ and $\varepsilon \to 0$.

After a certain identification of the eigenfunctions with pairs of functions, and a certain re-scaling of these functions in the suitable Sobolev spaces, we prove that both problems have a common *limiting eigenvalue problem* which is problem (2.15)–(2.16) in the space product $H_0^1(\Omega) \times H^1(\Gamma)$. In addition, we prove that when we set the small parameter $\delta_\varepsilon = \varepsilon^{t-1}$, the eigenelements of (1.1) provide an alternative approach for the eigenelements of (1.2) to those of the limiting problem. The approach is better for $t \in (1, 3)$.

Note that this is the first work where both the asymptotic behavior for the eigenelements of (1.1) and (1.2), and their connection are obtained. This connection can be important from a numerical viewpoint, since we deal with a fixed domain of reference avoiding computations in the thin band (cf. Figs. 1 and 2).

Let Ω be a bounded domain of $\mathbb{R}^2$ with a smooth boundary Γ and let (ν, τ) be the natural orthogonal curvilinear coordinates in a neighborhood of Γ: τ is the arc length and ν the distance along the outer normal vector to Γ. Let also ℓ denote the length of Γ and $\varkappa(\tau)$ the curvature of the curve Γ at the point τ. We assume that the domain Ω is surrounded by the thin band $\omega_\varepsilon = \{x\colon\ 0 < \nu < \varepsilon h(\tau)\}$ where $\varepsilon > 0$ is a small parameter and h is a strictly positive function of the τ-variable, ℓ-periodic, $h \in C^\infty(\mathbb{S}_\ell)$ where $\mathbb{S}_\ell$ stands for the circumference of length ℓ. Let Ω_ε be the

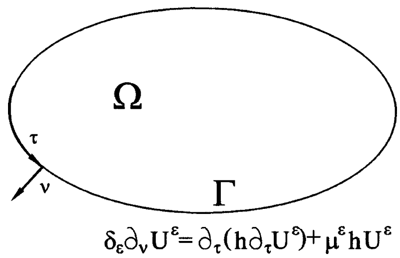

Fig. 1. Boundary condition for the Wentzell spectral problem.

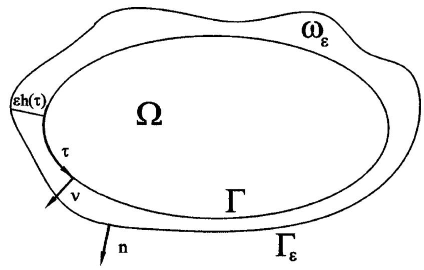

Fig. 2. Geometry of Ω_ε.

domain $\Omega_\varepsilon = \Omega \cup \omega_\varepsilon \cup \Gamma$ and $\Gamma_\varepsilon = \{x\colon\ \nu = \varepsilon h(\tau)\}$ the boundary of Ω_ε (see Fig. 2). Let δ_ε be a small parameter, $\delta_\varepsilon \to 0$ as $\varepsilon \to 0$.

We consider the following two parameter-dependent spectral problems.

First, the *Wentzell spectral problem*, with the small parameter δ_ε multiplying the normal derivative on the boundary condition:

$$\begin{cases} -A\Delta_x \mathbf{U}^\varepsilon = \mu^\varepsilon \mathbf{U}^\varepsilon & \text{in } \Omega, \\ \delta_\varepsilon A \partial_\nu \mathbf{U}^\varepsilon = \mu^\varepsilon h \mathbf{U}^\varepsilon + a\partial_\tau(h\partial_\tau \mathbf{U}^\varepsilon) & \text{on } \Gamma. \end{cases} \tag{1.1}$$

Second, the Neumann problem in Ω_ε for a second order differential operator with piecewise constant coefficients:

$$\begin{cases} -A\Delta_x U^\varepsilon = \lambda^\varepsilon U^\varepsilon & \text{in } \Omega, \\ -a\varepsilon^{-t}\Delta_x u^\varepsilon = \lambda^\varepsilon \varepsilon^{-t} u^\varepsilon & \text{in } \omega_\varepsilon, \\ U^\varepsilon = u^\varepsilon & \text{on } \Gamma, \\ A\partial_\nu U^\varepsilon = a\varepsilon^{-t}\partial_\nu u^\varepsilon & \text{on } \Gamma, \\ a\varepsilon^{-t}\partial_n u^\varepsilon = 0 & \text{on } \Gamma_\varepsilon. \end{cases} \tag{1.2}$$

In both problems, A and a are two positive constants while ∂_τ stands for the tangential derivative along Γ, and ∂_ν and ∂_n denote the derivatives along the outward normal vectors ν and n to the curves Γ and Γ_ε, respectively. Meanwhile, the parameter $\delta = \delta_\varepsilon$ in (1.1), in principle, can be any small parameter, $\delta \to 0$ independently from ε; we set $\delta = \delta_\varepsilon$ for simplicity in further notations, but it should be emphasized that only when comparing problems (1.1) and (1.2) does δ_ε depend on ε in a precise way (cf. (1.4)). In (1.2), t is a parameter which we set at $t > 1$.

Note that the spectral parameter μ^ε in (1.1) appears both in the equation in Ω and in the boundary condition on Γ, while the spectral parameter λ^ε in (1.2) appears in the equations in the ε-dependent domain Ω_ε, which "approaches" Ω as $\varepsilon \to 0$.

In this paper, we study the asymptotic behavior of the eigenelements $(\mu^\varepsilon, \mathbf{U}^\varepsilon)$ of problem (1.1) and $(\lambda^\varepsilon, \{U^\varepsilon, u^\varepsilon\})$ of problem (1.2) for $t > 1$. For proofs, without any loss of generality, we set $a = A = 1$.

As has already been pointed out in [2], problem (1.2) is of interest, for instance, in *reinforcement problems*. Recall that the parameter t reflects both the relative stiffness and dead-weight of the band respectively in mechanical problems, i.e., increasing t makes the band ω_ε both stiffer and heavier. Also, problem (1.2) is of interest in vibrations of two-phases systems in fluid mechanics.

As regards the Wentzell spectral problem, namely problem (1.1), it should be mentioned that boundary conditions involving an elliptic operator in the tangential variables along with the normal derivative have been stated originally by A.D. Wentzell (cf. [13]) in connection with diffusion problems. For a mechanical interpretation of these boundary conditions related to the vibrations of a "string-membrane" system see [5]. We also refer to [5] for further references on the problem without small parameters.

A characterization of the limiting problem, as $\varepsilon \to 0$, for the eigenelements of (1.2), has been obtained in [2] by means of asymptotic expansions. In fact, in [2], we give a characterization of the limiting problems for the eigenelements of several spectral problems: namely, Eqs. $(1.2)_1$, $(1.2)_3$–$(1.2)_5$, and

$$-a\varepsilon^{-t}\Delta_x u^\varepsilon = \lambda^\varepsilon \varepsilon^{-t-m} u^\varepsilon \quad \text{in } \omega_\varepsilon, \tag{1.3}$$

for different values of t and m, provided that $t \geqslant 0$ and $t + m \geqslant 0$, and either $t > 0$ or $t + m > 0$; the eigenvalues always being in the range of the low frequencies. Also in [2] sharp bounds for convergence rates of the eigenvalues and eigenfunctions in the case where $t = 1$ and $m = 0$ have been obtained by using the so-called *inverse-direct reduction procedure* (see Section 2.1). No justification for the asymptotic expansions for the eigenelements of (1.2) when $t > 1$ has been presented in [2].

In that connection, only the leading terms of the above mentioned asymptotic expansions for problem (1.2) have been considered in [2], and several questions remained about the asymptotic behavior of the eigenelements as $\varepsilon \to 0$. For instance, as we make it clear in this paper, new asymptotic expansions for the eigenfunctions in ω_ε (see (5.6) and (5.17)), coming from (2.4), lead us to the convergence.

On the other hand, it should be noted that, as outlined in [2], in problem $(1.2)_1$, $(1.2)_3$–$(1.2)_5$, (1.3), out of all the possible choices of t and m, the cases where $t > 1$ and $m \leqslant 0$ are the most conflictive cases. In the present paper, we consider one of these conflictive cases, namely $t > 1$ and $m = 0$ (cf. Remark 6.1), and we study the asymptotic behavior of the eigenvalues λ^ε and the eigenfunctions $\{U^\varepsilon, u^\varepsilon\}$ of problem (1.2) providing the justification of the asymptotic expansions and obtaining sharp bounds for convergence rates. In addition, we establish a connection of the eigenelements of (1.2) with those of (1.1) in the case, where

$$\delta_\varepsilon = \varepsilon^{t-1}. \tag{1.4}$$

Apart from this connection, we also show that both problems, (1.2) for $t > 1$, and (1.1), have a common limiting eigenvalue problem, which is (2.15)–(2.16) in the space $H_0^1(\Omega) \times H^1(\Gamma)$. The form in which the eigenelements of problem (1.1) ((1.2), respectively) are approached by those of (2.15)–(2.16) is stated in Sections 3 and 4 (5 and 6, respectively). It can be noticed that an asymptotic dissociation in two spectral problems on Ω and Γ arises.

The structure of the paper is as follows. In Section 2 we introduce the weak formulation of problems (1.1) and (1.2) and that of the limiting problem (2.15)–(2.16). We also introduce some notations used throughout the paper, and, for the sake of completeness, we state certain known results from the spectral perturbation theory useful for our approach.

Section 3 contains asymptotic expansions for the eigenelements of (1.1), the leading terms of these expansions (see (3.1)–(3.3)) being determined via the eigenelements of the limiting problem. We also obtain the higher order terms in the expansions and provide the method to derive the whole asymptotic series even though the limiting eigenvalue is a multiple eigenvalue of (2.15) or (2.16) or is an eigenvalue of both problems simultaneously of any multiplicity (the so-called *resonant case*). Note that a re-scaling of the eigenfunctions on the boundary is essential in order to convert the parameter dependent problem into another (cf. (2.4)) which allows the whole expansion to be obtained.

In Section 4 we justify the formal asymptotics derived in Section 3. Specifying, the convergence of the eigenelements of (1.1) towards those of (2.15)–(2.16) with conservation of the multiplicity is obtained along with precise estimates for the discrepancies of these eigenelements. This is stated in Theorems 4.1–4.2. Theorems 4.3–4.7 of this section show how to improve progressively these estimates by adding the higher order terms of the asymptotic expansions; that is, the results provide correcting terms for the eigenelements of (1.1) (cf. Remark 4.2).

The same idea for the eigenelements of (1.2) is used in Sections 5 and 6. In Section 5 we provide certain asymptotic expansions for the eigenelements of (1.2) (cf. (5.1)–(5.3) or (5.4)–(5.6)) which lead to the same limiting spectra, which is the union of the spectra of (2.15) and (2.16), but now the limiting problem (5.7) is non-selfadjoint, since we prove that associated functions can appear (see Proposition 5.1). This fact, which, at first sight, seems to be in contradiction with the fact that the initial problem (1.2) is self-adjoint, has already been noticed in [2] without any proof. Here, considering a particular t as a sample, we make it clear how this fact affects the asymptotic expansions. In addition, we prove that again a re-scaling of the eigenfunctions in ω_ε avoids the non self-adjoint problem. This is justified in Section 6 (cf. Theorem 6.1).

In Section 6.1, we compare the spectra of (1.2) and (1.1). We emphasize that, in the case where $\delta_\varepsilon = \varepsilon^{t-1}$, the results obtained allow us to assert that the eigenelements of (1.1) always provide an alternative approach to those of (1.2) (cf. Remark A.1) which is in fact a better approach for $t \in (1, 3)$. We also obtain important bounds for the discrepancies between the eigenelements of both problems.

Finally, Appendix A contains bounds for the discrepancies between the eigenelements of problem (1.1) ((1.2), respectively) and the limiting problem which improve bounds obtained in the previous sections (cf. Remark 4.1). The proofs, which are avoided for brevity, rely on the direct and inverse reduction method (see Section 2.1).

Let us note that asymptotics for vibrating systems containing a stiff region ω, ω independent of ε, have been considered by several authors over the last decades using very different techniques (cf. [9,12] for references). Vibrating systems with concentrated masses have also been widely approached in the literature. These studies consider the asymptotic behavior of the vibrations of systems (membranes or bodies) that contain concentrated masses along curves or masses concentrated at certain points (see [4,2] for an extensive bibliography on the subject). We refer to [2] as the only paper which addresses asymptotics for stiff spectral problems, with very large density and stiffness constants appearing simultaneously in a thin region. Let us mention [7] for different singularly perturbed spectral problems where it is made clear that an approach via a spectral problem in a domain with a regular perturbation of the boundary can be more suitable than the approach using the corresponding limiting spectral problems.

We emphasize that the problems, results and techniques in this paper are different from those in previous papers: Firstly, this is the first study of asymptotics for (1.1) in the literature of applied mathematics. Secondly, as regards problem (1.2), we consider convergences for different values of the parameter t than [2] and different asymptotic expansions. In addition, for both problems we obtain the convergence of the spectrum towards that of the limiting problem and provide asymptotic and uniform bounds for convergence rates depending on ε and the eigenvalue number.

Also, certain correcting terms for the eigenelements improving the above bounds are obtained. Finally, we provide an alternative approach for the eigenelements of (1.2) via the eigenelements of (1.1).

2. Preliminaries

We introduce the Hilbert spaces, weak formulations, notations and results of the spectral perturbation theory for further use. Throughout the paper, for sufficiently smooth functions V defined in a neighborhood of Γ, we refer to $V(\nu,\tau)$ as the function $V(x)$ written in curvilinear coordinates, and, if no confusion arises, we do not distinguish between a point τ on the boundary Γ and its coordinate along Γ.

Let us introduce the functional space $H^{1,1}(\Omega,\Gamma)$ as the completion of $C^\infty(\overline{\Omega})$ with respect to the norm

$$\|W\|_{H^{1,1}(\Omega,\Gamma)} = \left(\|W\|^2_{H^1(\Omega)} + \|W\|^2_{H^1(\Gamma)}\right)^{1/2}.$$

The weak formulation of problem (1.1) is: Find μ^ε and $\mathbf{U}^\varepsilon \in H^{1,1}(\Omega,\Gamma)$, $\mathbf{U}^\varepsilon \neq 0$, satisfying:

$$A\int_\Omega \nabla\mathbf{U}^\varepsilon\cdot\nabla W\,\mathrm{d}x + \delta_\varepsilon^{-1}a\int_\Gamma h\partial_\tau\mathbf{U}^\varepsilon\partial_\tau W\,\mathrm{d}\tau = \mu^\varepsilon\left[\int_\Omega \mathbf{U}^\varepsilon W\,\mathrm{d}x + \delta_\varepsilon^{-1}\int_\Gamma h\mathbf{U}^\varepsilon W\,\mathrm{d}\tau\right] \quad \forall W\in H^{1,1}(\Omega,\Gamma). \tag{2.1}$$

Its eigenelements are $(\mu^\varepsilon,\mathbf{U}^\varepsilon)$, where $((\mu^\varepsilon+1)^{-1},\mathbf{U}^\varepsilon)$ are the eigenelements of the positive, symmetric and compact operator $\mathcal{B}_\varepsilon$ on $H^{1,1}(\Omega,\Gamma)$ defined by:

$$(\mathcal{B}_\varepsilon U, V) = \int_\Omega UV\,\mathrm{d}x + \delta_\varepsilon^{-1}\int_\Gamma hUV\,\mathrm{d}\tau \quad \forall U,V\in H^{1,1}(\Omega,\Gamma).$$

For fixed ε, let

$$0=\mu_0^\varepsilon<\mu_1^\varepsilon\leqslant\mu_2^\varepsilon\leqslant\cdots\leqslant\mu_k^\varepsilon\leqslant\cdots\xrightarrow{k\to\infty}\infty,$$

include the eigenvalues of (2.1) with the usual convention of repeated eigenvalues. The corresponding eigenfunctions $\{\mathbf{U}_k^\varepsilon\}_{k=0}^\infty$ can be subject to the orthogonality condition,

$$\int_\Omega \mathbf{U}_k^\varepsilon\mathbf{U}_l^\varepsilon\,\mathrm{d}x + \delta_\varepsilon^{-1}\int_\Gamma h\mathbf{U}_k^\varepsilon\mathbf{U}_l^\varepsilon\,\mathrm{d}\tau = \delta_{k,l}\left(\mu_k^\varepsilon+1\right)^{-1}, \tag{2.2}$$

and form a basis in $H^{1,1}(\Omega,\Gamma)$. Here and in the sequel $\delta_{k,l}$ denotes the Kronecker symbol. See also [5] for another mathematical study on a Wentzell spectral problem of the type (1.1) without small parameter.

Using the minimax principle,

$$\mu_k^\varepsilon = \min_{\substack{E_k\subset H^{1,1}(\Omega,\Gamma)\\ \dim E_k=k+1}}\ \max_{\substack{V\in E_k\\ V\neq 0}} \frac{A\int_\Omega|\nabla_x V|^2\,\mathrm{d}x + a\delta_\varepsilon^{-1}\int_\Gamma h|\partial_\tau V|^2\,\mathrm{d}\tau}{\int_\Omega|V|^2\,\mathrm{d}x + \delta_\varepsilon^{-1}\int_\Gamma h|V|^2\,\mathrm{d}\tau},$$

where the minimum is taken over all the subspaces $E_k\subset H^{1,1}(\Omega,\Gamma)$ with $\dim E_k=k+1$, and considering the particular subspace $E_k^*=[V_0,\ldots,V_k]$, where $\{V_i\}_{i=0}^k$ is the set of eigenfunctions of the Dirichlet problem (2.15) associated with the $(k+1)$th first eigenvalues, we have a first estimate for the eigenvalues μ^ε of (2.1),

$$\mu_k^\varepsilon\leqslant\Lambda_k,\quad k=1,2,3,\ldots, \tag{2.3}$$

where Λ_k denotes the $(k+1)$th eigenvalue of the Dirichlet problem (2.15).

On the other hand, considering the normalization condition (2.2), we can introduce artificially a new unknown defined by $\mathbf{w}^\varepsilon(\tau)=\delta_\varepsilon^{-1/2}\mathbf{U}^\varepsilon(0,\tau)$ for $\tau\in\mathbb{S}_\ell$. Then, problem (1.1) is equivalent to:

$$\begin{cases} -A\Delta_x\mathbf{U}^\varepsilon=\mu^\varepsilon\mathbf{U}^\varepsilon & \text{in } \Omega,\\ \mathbf{U}^\varepsilon=\delta_\varepsilon^{1/2}\mathbf{w}^\varepsilon & \text{on } \Gamma,\\ a\partial_\tau(h\partial_\tau\mathbf{w}^\varepsilon)+\mu^\varepsilon h\mathbf{w}^\varepsilon=\delta_\varepsilon^{1/2}A\partial_\nu\mathbf{U}^\varepsilon & \text{on } \Gamma. \end{cases} \tag{2.4}$$

Let us denote by $\mathcal{H}_\varepsilon$ the subspace of $H^1(\Omega)\times H^1(\Gamma)$ defined by:

$$\mathcal{H}_\varepsilon=\left\{(U,u)\in H^1(\Omega)\times H^1(\Gamma)\mid U=\delta_\varepsilon^{1/2}u \text{ on } \Gamma\right\}, \tag{2.5}$$

equipped with the scalar product in $H^1(\Omega) \times H^1(\Gamma)$,

$$\big(\!\big((U,u),(W,w)\big)\!\big) = A\int_\Omega \nabla U\cdot\nabla W\,\mathrm{d}x + a\int_\Gamma h\partial_\tau u\partial_\tau w\,\mathrm{d}\tau + \int_\Omega UW\,\mathrm{d}x + \int_\Gamma huw\,\mathrm{d}\tau, \tag{2.6}$$

and $|||(\cdot,\cdot)|||$ denotes the associated norm. After identifying the elements of $H^{1,1}(\Omega,\Gamma)$ with pairs of functions $(U,u)\in H^1(\Omega)\times H^1(\Gamma)$ we can write the weak formulation of (2.4) as follows: Find μ^ε and $(\mathbf{U}^\varepsilon,\mathbf{w}^\varepsilon)\in\mathcal{H}_\varepsilon$, $(\mathbf{U}^\varepsilon,\mathbf{w}^\varepsilon)\neq 0$, satisfying,

$$\big(\!\big(\big(\mathbf{U}^\varepsilon,\mathbf{w}^\varepsilon\big),(W,w)\big)\!\big) = \big(\mu^\varepsilon+1\big)\bigg[\int_\Omega \mathbf{U}^\varepsilon W\,\mathrm{d}x + \int_\Gamma h\mathbf{w}^\varepsilon w\,\mathrm{d}\tau\bigg] \quad \forall (W,w)\in\mathcal{H}_\varepsilon. \tag{2.7}$$

In addition, condition (2.2) for the eigenfunctions amounts to

$$\big|\big|\big|\big(\mathbf{U}_k^\varepsilon,\mathbf{w}_k^\varepsilon\big)\big|\big|\big| = 1 \quad\text{and}\quad \big(\!\big(\big(\mathbf{U}_k^\varepsilon,\mathbf{w}_k^\varepsilon\big),\big(\mathbf{U}_l^\varepsilon,\mathbf{w}_l^\varepsilon\big)\big)\!\big) = \delta_{k,l},\quad k,l=0,1,2,\ldots. \tag{2.8}$$

As regards problem (1.2), we can identify the function u_ε in $L^2(\Omega_\varepsilon)$ ($H^1(\Omega_\varepsilon)$, respectively) with the pair of functions $\{U^\varepsilon,u^\varepsilon\}$ where U^ε stands for the restriction of u_ε into Ω and u^ε for the restriction of u_ε into ω_ε. The variational formulation of (1.2) in the couple of spaces $H^1(\Omega_\varepsilon)\subset L^2(\Omega_\varepsilon)$, for any t, has been introduced in [2], where also certain bounds for the eigenvalues have been obtained.

In this paper we consider $t>1$, and, for each fixed ε, we denote by:

$$0=\lambda_0^\varepsilon<\lambda_1^\varepsilon\leqslant\lambda_2^\varepsilon\leqslant\cdots\leqslant\lambda_k^\varepsilon\leqslant\cdots\xrightarrow{k\to\infty}\infty,$$

the sequence of eigenvalues repeated according to their multiplicities. They also satisfy (see [2]),

$$\lambda_k^\varepsilon\leqslant\Lambda_k,\quad k=1,2,\ldots. \tag{2.9}$$

The corresponding eigenfunctions $\{\{U_k^\varepsilon,u_k^\varepsilon\}\}_{k=0}^\infty$ are subject to the orthonormalization condition:

$$\int_\Omega U_k^\varepsilon U_l^\varepsilon\,\mathrm{d}x + \varepsilon^{-t}\int_{\omega_\varepsilon} u_k^\varepsilon u_l^\varepsilon\,\mathrm{d}x = \delta_{k,l}\big(\lambda_k^\varepsilon+1\big)^{-1}. \tag{2.10}$$

Also, the elements $\{U^\varepsilon,u^\varepsilon\}\in H^1(\Omega_\varepsilon)$ can be identified with the pairs $(U^\varepsilon,w^\varepsilon)\in H^1(\Omega)\times H^1(\omega_\varepsilon)$ where $w^\varepsilon=\varepsilon^{(1-t)/2}u^\varepsilon$ and this leads us to a new weak formulation of (1.2) different from that in [2]. To this end, it proves useful to introduce a new space $\mathcal{V}_\varepsilon$, which is the subspace of $H^1(\Omega)\times H^1(\omega_\varepsilon)$ defined as

$$\mathcal{V}_\varepsilon=\big\{(U,u)\in H^1(\Omega)\times H^1(\omega_\varepsilon)\mid U=\varepsilon^{\frac{t-1}{2}}u \text{ on } \Gamma\big\}, \tag{2.11}$$

equipped with the scalar product in $H^1(\Omega)\times H^1(\omega_\varepsilon)$:

$$\big((U,u),(W,w)\big)_\varepsilon = A\int_\Omega\nabla U\cdot\nabla W\,\mathrm{d}x + \frac{a}{\varepsilon}\int_{\omega_\varepsilon}\nabla_x u\cdot\nabla_x w\,\mathrm{d}x + \int_\Omega UW\,\mathrm{d}x + \frac{1}{\varepsilon}\int_{\omega_\varepsilon} uw\,\mathrm{d}x. \tag{2.12}$$

Let $\|(\cdot,\cdot)\|_\varepsilon$ denote the associated norm in $\mathcal{V}_\varepsilon$.

Now, the variational formulation of (1.2), with $t>1$, can be written in $\mathcal{V}_\varepsilon$ as follows: Find λ^ε and $(U^\varepsilon,w^\varepsilon)\in\mathcal{V}_\varepsilon$, $(U^\varepsilon,w^\varepsilon)\neq 0$, satisfying

$$\big(\big(U^\varepsilon,w^\varepsilon\big),(W,w)\big)_\varepsilon = \big(\lambda^\varepsilon+1\big)\bigg[\int_\Omega U^\varepsilon W\,\mathrm{d}x + \frac{1}{\varepsilon}\int_{\omega_\varepsilon} w^\varepsilon w\,\mathrm{d}x\bigg]\quad\forall(W,w)\in\mathcal{V}_\varepsilon. \tag{2.13}$$

Then, normalization (2.10) for the eigenfunctions, reads

$$\big\|\big(U_k^\varepsilon,w_k^\varepsilon\big)\big\|_\varepsilon=1 \quad\text{and}\quad \big(\big(U_k^\varepsilon,w_k^\varepsilon\big),\big(U_l^\varepsilon,w_l^\varepsilon\big)\big)_\varepsilon=\delta_{k,l},\quad k,l=0,1,2,\ldots, \tag{2.14}$$

and, obviously, the eigenelements of (1.2) for $t>1$, $(\lambda^\varepsilon,\{U^\varepsilon,u^\varepsilon\})\in\mathbb{R}\times H^1(\Omega_\varepsilon)$ give us those of (2.13) $(\lambda^\varepsilon,(U^\varepsilon,\varepsilon^{\frac{1-t}{2}}u^\varepsilon))\in\mathbb{R}\times\mathcal{V}_\varepsilon$.

Let us consider the Dirichlet spectral problem in Ω, namely,

$$\begin{cases} -A\Delta_x V = \mu V & \text{in } \Omega, \\ V = 0 & \text{on } \Gamma, \end{cases} \tag{2.15}$$

and the periodic spectral Sturm–Liouville problem in the τ variable, namely,

$$a\partial_\tau(h\partial_\tau W) + h\mu W = 0 \quad \text{on } \Gamma. \tag{2.16}$$

As is well known, each problem, (2.15) and (2.16) with spectral parameter μ, has a well-determined real discrete spectrum, and these spectra can intersect depending on the geometry of Ω, constants a and A and function $h(\tau)$. We denote by $\sigma(P_\Omega)$ and $\sigma(P_\Gamma)$ the respective spectra of (2.15) and (2.16). Let $\kappa_\Omega(\mu)$ ($\kappa_\Gamma(\mu)$, respectively) denote the multiplicity of μ as an eigenvalue of problem (2.15) ((2.16), respectively), with the assumption that the multiplicity is zero if μ is not an eigenvalue.

The problem (2.15)–(2.16) in the space $H_0^1(\Omega) \times H^1(\Gamma)$, which also has a discrete spectrum, is referred to as the *limiting problem*. Its eigenelements are $(\mu, (V, W)) \in \mathbb{R} \times (H_0^1(\Omega) \times H^1(\Gamma))$ such that (μ, V) is an eigenelement of (2.15) or (μ, W) is an eigenelement of (2.16). Obviously, the multiplicity of μ as an eigenvalue of (2.15)–(2.16) is $\kappa_\Omega(\mu) + \kappa_\Gamma(\mu)$.

The weak formulation of problem (2.15)–(2.16) in the space $H_0^1(\Omega) \times H^1(\Gamma)$ is: Find μ and $(V_0, w_0) \in H_0^1(\Omega) \times H^1(\Gamma)$, $(V_0, w_0) \neq 0$, satisfying,

$$\big(\big((V_0, w_0), (W, w)\big)\big) = \big(\mu^0 + 1\big)\left[\int_\Omega V_0 W \, dx + \int_\Gamma h w_0 w \, d\tau\right] \quad \forall (W, w) \in H_0^1(\Omega) \times H^1(\Gamma), \tag{2.17}$$

where $((\cdot,\cdot))$ denotes the scalar product in $H_0^1(\Omega) \times H^1(\Gamma)$ given by (2.6).

Let us consider,

$$0 = \mu_0^0 < \mu_1^0 \leqslant \mu_2^0 \leqslant \cdots \leqslant \mu_k^0 \leqslant \cdots \xrightarrow{k\to\infty} \infty,$$

the eigenvalues of (2.17) repeated according to their multiplicities. Let $\{(V_k, w_k)\}_{k=0}^{\infty}$ be the corresponding eigenfunctions, which are subject to the orthogonality condition:

$$\int_\Omega V_k V_l \, dx + \int_\Gamma h w_k w_l \, d\tau = \delta_{k,l}\big(\mu_k^0 + 1\big)^{-1}, \tag{2.18}$$

and form a basis in $H_0^1(\Omega) \times H^1(\Gamma)$.

Next, in order to obtain asymptotic expansions for the eigenpairs of problems (1.1) and (1.2), in Sections 3 and 5 respectively, we introduce certain notations for further use.

Since a boundary layer phenomenon appears in a neighborhood of Γ, we introduce the so-called *rapid variable* $\zeta = \nu/\varepsilon$. ζ is also called the *local variable* and transforms the thin domain ω_ε into a band of length ℓ and width O(1), namely, $\{\nu \in [0, \varepsilon h(\tau)), \tau \in \mathbb{S}_\ell\}$ into $\{\zeta \in [0, h(\tau)), \tau \in \mathbb{S}_\ell\}$. Writing the Laplace operator in curvilinear coordinates (ν, τ),

$$\Delta_x = K(\nu,\tau)^{-1}\partial_\nu\big(K(\nu,\tau)\partial_\nu\big) + K(\nu,\tau)^{-1}\partial_\tau\big(K(\nu,\tau)^{-1}\partial_\tau\big),$$

where $K(\nu,\tau) = 1 + \nu\varkappa(\tau)$ and $\varkappa(\tau)$ is the curvature of the curve Γ at the point τ, we introduce the change to the rapid variable ζ. Then, since $(1+\varepsilon\zeta\varkappa)^{-1} = \sum_{i=0}^{\infty}(-\varepsilon\zeta\varkappa)^i$ for sufficiently small ε, we have:

$$\Delta_{\zeta,\tau} = \varepsilon^{-2}\partial_\zeta^2 + \varepsilon^{-1}\varkappa(\tau)\partial_\zeta + \varepsilon^0\big(\partial_\tau^2 - \varkappa(\tau)^2\zeta\partial_\zeta\big) + \varepsilon\big(\varkappa(\tau)^3\zeta^2\partial_\zeta - 2\varkappa(\tau)\zeta\partial_\tau^2 - \varkappa'(\tau)\zeta\partial_\tau\big) + \cdots, \tag{2.19}$$

where (as in the sequel) we denote by dots further asymptotic terms of different powers of ε which in general we do not use to derive our results in the paper.

According to the definition of Γ_ε and the representation of the gradient in the curvilinear coordinates (ν, τ), the normal derivative at the boundary Γ_ε reads,

$$\partial_n = \big(1 + \varepsilon^2 K(\nu,\tau)^{-2}h'(\tau)^2\big)^{-1/2}\big(\partial_\nu - \varepsilon h'(\tau)K(\nu,\tau)^{-2}\partial_\tau\big),$$

where $h'(\tau)$ denotes the derivative $\partial_\tau h(\tau)$. Therefore,

$$\big(1 + \varepsilon^2 K(\nu,\tau)^{-2}h'(\tau)^2\big)^{1/2}\partial_n = \varepsilon^{-1}\partial_\zeta - \varepsilon h'(\tau)\partial_\tau + \cdots. \tag{2.20}$$

On the other hand, on account of the continuity of functions $h(\tau)$ and curvature $\varkappa(\tau)$, for sufficiently small $d > 0$, there exist constants c, C_1, C_2 and C_3 independent of ε such that

$$0 < c < K(\nu, \tau) < C_1 \quad \forall \nu \in [-d, d],\ \tau \in \mathbb{S}_\ell, \tag{2.21}$$

$$\bigl|1 - K(\nu, \tau)\bigr| \leqslant C_2\varepsilon \quad \text{and} \quad \bigl|1 - K(\nu, \tau)^{-1}\bigr| \leqslant C_3\varepsilon \quad \forall \nu \in \bigl[0, \varepsilon h(\tau)\bigr],\ \tau \in \mathbb{S}_\ell. \tag{2.22}$$

As above, throughout all the paper, if no confusion arises, C, c, $\widetilde{C}$, C_*, $C_0, C_1, C_2, \ldots,$ and C_k denote constants independent of ε.

2.1. Background for convergences

For the sake of completeness, we introduce here three known results that we shall use to prove the convergence and to obtain precise bounds for discrepancies in Sections 4, 6 and Appendix A.

The first one is a classical result on "almost eigenvalues and eigenvectors" from the spectral perturbation theory (cf. Lemma 2.1); the second one is an algebraic result which provides information on the total multiplicity of the eigenvalues of the ε-dependent problem in certain intervals (cf. Lemma 2.2). Both results are used in the *inverse-direct reduction procedure*. This procedure is a general method in singularly perturbed spectral problems which is intended both to justify asymptotic expansions for eigenvalues and eigenfunctions and to show convergence while obtaining the explicit dependence of convergence rates on the perturbation parameter and the eigenvalue number (cf. [6,9,2] for an extensive explanation of the method and for further references on the subject).

Finally, Lemma 2.3 is *the spectral convergence theorem* for positive, symmetric and compact operators on parameter-dependent Hilbert spaces. In order to state this lemma we use the following notation: given two Hilbert spaces H_1 and H_2, we denote by $\mathcal{L}(H_1, H_2)$ ($\mathcal{L}(H_1)$ respectively) the space of the linear continuous operators from H_1 into H_2 (into H_1 respectively).

Lemma 2.1. *Let $A : H \to H$ be a linear, self-adjoint, positive and compact operator on a separable Hilbert space H. Let $u \in H$, with $\|u\|_H = 1$ and λ, $r > 0$ such that $\|Au - \lambda u\|_H \leqslant r$. Then, there exists an eigenvalue λ_i of the operator A satisfying the inequality $|\lambda - \lambda_i| \leqslant r$. Moreover, for any $r^* > r$ there is $u^* \in H$, with $\|u^*\|_H = 1$, u^* belonging to the eigenspace associated with all the eigenvalues of the operator A lying on the segment $[\lambda - r^*, \lambda + r^*]$ and such that*

$$\bigl\|u - u^*\bigr\|_H \leqslant \frac{2r}{r^*}.$$

Lemma 2.2. *Let H be a Hilbert space with the scalar product $(\cdot,\cdot)_H$. Let $w^1, \ldots, w^n \in H$ and $W^1, \ldots, W^N \in H$ fulfill the following properties*:

(a) *for $i, j = 1, \ldots, n$, $(w^i, w^j)_H = \delta_{i,j}$;*
(b) *for $i, j = 1, \ldots, n$, $\|W^j\|_H = 1$, $|(W^i, W^j)_H - \delta_{i,j}| \leqslant \varrho$, where ϱ is a positive constant independent of i, j;*
(c) *for $j = 1, \ldots, N$, there exist constants $\{a_q^j\}_{q=1}^n$ such that $\|W^j - \sum_{q=1}^n a_q^j w^q\|_H \leqslant \sigma$, where σ is a positive constant independent of j.*

Then, under the condition $(\min\{n, N\} + 1)(\varrho + (2 + \sigma)\sigma) < 1$, the inequality $N \leqslant n$ holds. In addition, in the case where $n = N$, the condition $n(\varrho + (2 + \sigma)\sigma) < 1$ ensures the existence of the unitary matrix $\theta = (\theta_q^j)_{q,j=1,\ldots,n}$ such that for $j = 1, \ldots, n$,

$$\Biggl\|w^j - \sum_{q=1}^n \theta_q^j W^q\Biggr\|_H \leqslant n\bigl(\varrho + (3 + \sigma)\sigma\bigr).$$

Lemma 2.3. *Let H_ε and H_0 be two separable Hilbert spaces with the scalar products $(\cdot,\cdot)_\varepsilon$ and $(\cdot,\cdot)_0$ respectively. Let $A^\varepsilon \in \mathcal{L}(H_\varepsilon)$ and $A^0 \in \mathcal{L}(H_0)$. Let $\mathcal{W}$ be a subspace of H_0 such that $\mathrm{Im}(A^0) = \{v \mid v = A^0 u\colon\ u \in H_0\} \subset \mathcal{W}$. We assume that the following properties are satisfied*:

(a) *There exists an operator* $R^\varepsilon \in \mathcal{L}(H_0, H_\varepsilon)$ *and a constant* $a > 0$ *such that, for any* $f \in \mathcal{W}$, $\|R^\varepsilon f\|_\varepsilon$ *converge towards* $a\|f\|_0$ *as* $\varepsilon \to 0$.
(b) A^ε *and* A^0 *are positive, compact and self-adjoint operators on* H^ε *and* H^0 *respectively. Besides, the norms* $\|A^\varepsilon\|_{\mathcal{L}(H_\varepsilon)}$ *are bounded by a constant independent of* ε.
(c) *For any* $f \in \mathcal{W}$, $\|A^\varepsilon R^\varepsilon f - R^\varepsilon A^0 f\|_\varepsilon \xrightarrow{\varepsilon \to 0} 0$.
(d) *The family of operators* A^ε *is uniformly compact, i.e., for any sequence* f^ε *in* H_ε *such that* $\sup_\varepsilon \|f^\varepsilon\|_\varepsilon$ *is bounded by a constant independent of* ε, *we can extract a subsequence* $f^{\varepsilon'}$ *verifying* $\|A^{\varepsilon'} f^{\varepsilon'} - R^{\varepsilon'} w^0\|_{\varepsilon'} \to 0$, *as* $\varepsilon' \to 0$, *for certain* $w^0 \in \mathcal{W}$.

Let $\{\mu_i^\varepsilon\}_{i=1}^\infty$ *and* $\{\mu_i^0\}_{i=1}^\infty$ *be the sequences of the eigenvalues of* A^ε *and* A^0, *respectively, with the usual convention of repeated eigenvalues. Let* $\{w_i^\varepsilon\}_{i=1}^\infty$ *and* ($\{w_i^0\}_{i=1}^\infty$, *respectively*) *be the corresponding eigenfunctions in* H_ε *which are assumed to be orthonormal* (H_0, *respectively*).

Then, for each fixed k *there exists a constant* C_k *independent of* ε *and there is* $\varepsilon_k > 0$ *such that for* $\varepsilon \leqslant \varepsilon_k$,

$$\left|\mu_k^\varepsilon - \mu_k^0\right| \leqslant C_k \sup\left\|A^\varepsilon R^\varepsilon u - R^\varepsilon A^0 u\right\|_\varepsilon,$$

where the sup *is taken over all the functions* u *in the eigenspace associated with* μ_k^0, u *such that* $\|u\|_0 = 1$. *In addition, for any eigenvalue* μ_k^0 *of* A^0 *with multiplicity* s ($\mu_k^0 = \mu_{k+1}^0 = \cdots = \mu_{k+s-1}^0$), *and for any* w *eigenfunction associated with* μ_k^0, *with* $\|w\|_0 = 1$, *there exists* w^ε, w^ε *being a linear combination of eigenfunctions of* A^ε $\{w_j^\varepsilon\}_{j=k}^{j=k+s-1}$ *associated with* $\{\mu_j^\varepsilon\}_{j=k}^{j=k+s-1}$, *such that*

$$\left\|w^\varepsilon - R^\varepsilon w\right\|_\varepsilon \leqslant M_k\left\|A^\varepsilon R^\varepsilon w - R^\varepsilon A^0 w\right\|_\varepsilon,$$

for a certain constant M_k *is independent of* ε.

We refer to Section III.1 in [10] for the proofs of Lemmas 2.1 and 2.3, and Section 7.1.4 in [6] for the proof of Lemma 2.2.

Throughout the paper, we use Lemma 2.3 to prove convergence of the spectrum with conservation of the multiplicity, and Lemmas 2.1 and 2.2 to obtain uniform bounds for convergence rates which are expressed in terms of the parameter ε, the eigenvalue number k and on properties of the spectrum of the limiting eigenvalue problem in an explicit form; that is, the rest of the constants appearing in the estimates do not depend on k and ε. Besides, the above mentioned properties of the limiting spectrum are related with the distance to the nearest eigenvalue and its multiplicity. In general, this explicit dependence cannot be detected with the theorems on spectral convergence, i.e., for instance, applying the result in Lemma 2.3 or other general results in [3]. These bounds for our problems are stated in Appendix A, and are obtained using the inverse-direct procedure. Nevertheless, since the method involves cumbersome computations, we avoid introducing their proofs here. Rather, the technique in this paper allows us to simplify computations: we combine certain tools of the method (cf. [2] for a related problem) with the spectral convergence theorem to obtain also precise asymptotic bounds for convergence rates for the eigenelements (see Remark 4.1 to compare).

3. Asymptotics for the Wentzell spectral problem

The aim of this section is to obtain the whole asymptotic series for the eigenelements of (2.4). In a first step we identify the limiting problems of (2.4) satisfied by the leading terms of the series, namely problems (2.15) and (2.16). Recall that $\sigma(P_\Omega)$ and $\sigma(P_\Gamma)$ denote the spectra of (2.15) and (2.16) respectively. In further steps we provide the method to obtain all the terms of the asymptotic expansions, depending on whether $\sigma(P_\Omega) \cap \sigma(P_\Gamma)$ is empty or not. The method also applies in the case where the eigenvalues of the limiting problems are not simple.

We consider an asymptotic expansion for the eigenvalues μ^ε and an expansion for the corresponding eigenfunctions $(\mathbf{U}^\varepsilon, \mathbf{w}^\varepsilon)$ in $\Omega \times \Gamma$ of the form:

$$\mu^\varepsilon = \mu^0 + \delta_\varepsilon^{1/2}\mu^{1/2} + \delta_\varepsilon \mu^1 + \cdots, \tag{3.1}$$

$$\mathbf{U}^\varepsilon(x) = V_0(x) + \delta_\varepsilon^{1/2} V_{1/2}(x) + \delta_\varepsilon V_1(x) + \cdots, \quad x \in \Omega, \tag{3.2}$$

$$\mathbf{w}^\varepsilon(\tau) = w_0(\tau) + \delta_\varepsilon^{1/2} w_{1/2}(\tau) + \delta_\varepsilon w_1(\tau) + \cdots, \quad \tau \in \mathbb{S}_\ell, \tag{3.3}$$

respectively, where $w_i(\tau)$ are ℓ-periodic functions in τ. Besides, we assume that the first term μ^0 in (3.1) can be 0 while at least one of the functions V_0 and w_0 in (3.2)–(3.3) are different from zero.

Note that the perturbation parameter in (2.4) is $\delta_\varepsilon^{1/2}$ and, on account of (2.3) and (2.8), we have chosen the different powers of $\delta_\varepsilon^{1/2}$ as the natural order functions in the asymptotic expansions.

We replace expansions (3.1)–(3.3) in problem (2.4) and collect coefficients of the same powers of δ_ε. In a first step, we have that the leading terms in (3.1) and (3.2), (μ^0, V_0), satisfy the Dirichlet problem in Ω (2.15), while the leading terms in (3.1) and (3.3), (μ^0, w_0), verify problem (2.16). Therefore, we have $\mu^0 \in \sigma(P_\Omega) \cup \sigma(P_\Gamma)$ and $V_0 = 0$ or $w_0 = 0$ in the case where $\mu^0 \notin \sigma(P_\Omega) \cap \sigma(P_\Gamma)$. Let us consider the three possibilities separately.

Case where $\boldsymbol{\mu^0 \in \sigma(P_\Omega)}$, with multiplicity $\boldsymbol{\kappa_\Omega(\mu^0) \geqslant 1}$, and $\boldsymbol{\mu^0 \notin \sigma(P_\Gamma)}$: *then, we prove that* $\mu^{1/2} = 0$ *in* (3.1) *while the asymptotic series* (3.1)–(3.3) *split into* $\kappa_\Omega(\mu^0)$ *branches.*

We consider μ^0 an eigenvalue of the Dirichlet problem (2.15) and denote its multiplicity $\kappa_\Omega(\mu^0) = q \geqslant 1$. Let $V^1, \dots, V^q$ be the corresponding eigenfunctions of (2.15) associated with μ^0, which are orthonormal in $L^2(\Omega)$, that is

$$\int_\Omega V^i V^j \, \mathrm{d}x = \delta_{i,j} \quad \text{for } i, j = 1, \dots, q. \tag{3.4}$$

Then,

$$V_0 = \alpha_1 V^1 + \cdots + \alpha_q V^q \tag{3.5}$$

where α_i are certain constants satisfying $\sum_{i=1}^q |\alpha_i|^2 > 0$, and $w_0 = 0$. In the following steps we fix the constants α_i which allow us to determine V_0.

Collecting coefficients of the following powers of δ_ε, in a second step, we obtain the problems:

$$\begin{cases} -\Delta_x V_{1/2} = \mu^0 V_{1/2} + \mu^{1/2} V_0 & \text{in } \Omega, \\ V_{1/2} = w_0 & \text{on } \Gamma, \end{cases} \tag{3.6}$$

$$\partial_\tau (h \partial_\tau w_{1/2}) + \mu^0 h w_{1/2} + \mu^{1/2} h w_0 = \partial_\nu V_0 \quad \text{on } \Gamma. \tag{3.7}$$

The compatibility conditions for the non-homogeneous Dirichlet problem (3.6), the orthogonality condition (3.4) and $w_0 = 0$ give us $0 = \int_\Gamma \partial_\nu V^i w_0 \, \mathrm{d}\tau = \mu^{1/2} \int_\Omega V_0 V^i \, \mathrm{d}x = \mu^{1/2} \alpha_i$ for $i = 1, \dots, q$ and, in consequence, $\mu^{1/2} = 0$ and $V_{1/2}$ is also an eigenfunction of (2.15). Let $V_{1/2} = \beta_1 V^1 + \cdots + \beta_q V^q$ with certain constants β_i to be determined.

Also, since $\mu^0 \notin \sigma(P_\Gamma)$, there exists an unique solution $w_{1/2}$ of (3.7). Indeed, because of (3.5), $w_{1/2}$ can be written as

$$w_{1/2} = \alpha_1 w_{1/2}^1 + \cdots + \alpha_q w_{1/2}^q \tag{3.8}$$

where, for $i = 1, \dots, q$, $w_{1/2}^i$ is the unique solution of

$$\partial_\tau \left(h \partial_\tau w_{1/2}^i \right) + \mu^0 h w_{1/2}^i = \partial_\nu V^i \quad \text{on } \Gamma. \tag{3.9}$$

In a third step, we have the problems:

$$\begin{cases} -\Delta_x V_1 = \mu^0 V_1 + \mu^{1/2} V_{1/2} + \mu^1 V_0 & \text{in } \Omega, \\ V_1 = w_{1/2} & \text{on } \Gamma, \end{cases} \tag{3.10}$$

$$\partial_\tau (h \partial_\tau w_1) + \mu^0 h w_1 + \mu^{1/2} h w_{1/2} + \mu^1 h w_0 = \partial_\nu V_{1/2} \quad \text{on } \Gamma. \tag{3.11}$$

Now, by virtue of (3.4), (3.5), (3.8) and the fact that $\mu^{1/2} = 0$, the compatibility conditions for the non-homogeneous Dirichlet problem (3.10) read:

$$\sum_{k=1}^q \alpha_k \int_\Gamma \partial_\nu V^i w_{1/2}^k \, \mathrm{d}\tau = \mu^1 \alpha_i \quad \text{for } i = 1, \dots, q.$$

Thus, we deduce that μ^1 is an eigenvalue of the matrix

$$\mathbf{M} = \left(\int_\Gamma \partial_\nu V^i w_{1/2}^k \, \mathrm{d}\tau \right)_{i,k=1,\ldots,q}, \tag{3.12}$$

and $\alpha = (\alpha_1, \ldots, \alpha_q)^{\mathrm{T}}$ is the corresponding eigenvector. From the definition of $w_{1/2}^i$, we obtain that the matrix $\mathbf{M}$ is symmetric and, consequently, it has q real eigenvalues μ_s^1 (with the usual convention of repeated eigenvalues), $s = 1, \ldots, q$; their corresponding eigenvectors $\alpha^s = (\alpha_1^s, \ldots, \alpha_q^s)^{\mathrm{T}}$ are orthogonal in $\mathbb{R}^q$, that is $\sum_{k=1}^q \alpha_k^s \alpha_k^i = \delta_{s,i}$ for $s, i = 1, \ldots, q$. Then, μ^ε splits into q branches from μ^0:

$$\mu^\varepsilon = \mu^0 + \delta_\varepsilon \mu_s^1 + \mathrm{o}(\delta_\varepsilon), \quad s = 1, \ldots, q,$$

with the corresponding eigenfunctions

$$\mathbf{U}^\varepsilon = \left(\alpha_1^s V^1 + \cdots + \alpha_q^s V^q\right) + \mathrm{o}(1) \quad \text{in } \Omega,\ s = 1, \ldots, q,$$
$$\mathbf{w}^\varepsilon = \delta_\varepsilon^{1/2}\left(\alpha_1^s w_{1/2}^1 + \cdots + \alpha_q^s w_{1/2}^q\right) + \mathrm{o}\left(\delta_\varepsilon^{1/2}\right) \quad \text{on } \Gamma,\ s = 1, \ldots, q.$$

We can continue the process and determine all terms of the expansions.

Case where $\boldsymbol{\mu^0 \in \sigma(P_\Gamma)}$, with multiplicity $\boldsymbol{\kappa_\Gamma(\mu^0) \geqslant 1}$, and $\boldsymbol{\mu^0 \notin \sigma(P_\Omega)}$: *then, we prove that $\mu^{1/2} = 0$ in* (3.1) *while the asymptotic series* (3.1)–(3.3) *split into $\kappa_\Gamma(\mu^0)$ branches.*

We consider μ^0 an eigenvalue of (2.16) and denote its multiplicity $\kappa_\Gamma(\mu^0) = p$, with $p = 1$ or 2. Let $w^1, \ldots, w^p$ be the corresponding eigenfunctions of (2.16) associated with μ^0, which are orthonormal in $L_h^2(\Gamma)$, that is

$$\int_\Gamma h w^i w^j \, \mathrm{d}\tau = \delta_{i,j} \quad \text{for } i, j = 1, \ldots, p. \tag{3.13}$$

Then, $V_0 = 0$ and

$$w_0 = d_1 w^1 + \cdots + d_p w^p \tag{3.14}$$

with d_j certain constants, $\sum_{j=1}^p |d_j|^2 > 0$, to be determined in the following steps.

In a second step, we have problems (3.6) and (3.7). In this case, the compatibility conditions for the non-homogeneous problem (3.7), the orthogonality condition (3.13) and $V_0 = 0$ allow us to assert that $\mu^{1/2} = 0$ and $w_{1/2}$ is also an eigenfunction of (2.16) associated with μ^0; let $w_{1/2}$ be $w_{1/2} = f_1 w^1 + \cdots + f_p w^p$ where f_j are certain constants to be determined.

In addition, since μ^0 is not an eigenvalue of problem (2.15), there exists an unique solution $V_{1/2}$ of (3.6), which can be written as

$$V_{1/2} = d_1 V_{1/2}^1 + \cdots + d_p V_{1/2}^p, \tag{3.15}$$

where, for $j = 1, \ldots, p$, $V_{1/2}^j$ is the unique solution of

$$\begin{cases} -\Delta_x V_{1/2}^j = \mu^0 V_{1/2}^j & \text{in } \Omega, \\ V_{1/2}^j = w^j & \text{on } \Gamma. \end{cases} \tag{3.16}$$

In the third step, we get problems (3.10) and (3.11). Now, by virtue of (3.13), (3.14), (3.15) and the fact that $\mu^{1/2} = 0$, the compatibility conditions for the non-homogeneous problem (3.11) read:

$$\sum_{k=1}^p d_k \int_\Gamma \partial_\nu V_{1/2}^j w^k \, \mathrm{d}\tau = \mu^1 d_j \quad \text{for } j = 1, \ldots, p,$$

and μ^1 is an eigenvalue of the matrix

$$\mathbf{N} = \left(\int_\Gamma \partial_\nu V_{1/2}^j w^k \, \mathrm{d}\tau \right)_{j,k=1,\ldots,p}, \tag{3.17}$$

and $d = (d_1, \dots, d_p)^{\mathrm{T}}$ is the corresponding eigenvector. From the definition of $V_{1/2}^j$ in (3.16), we verify that the matrix $\mathbf{N}$ is symmetric and, consequently, it has p real eigenvalues μ_t^1 (with the usual convention of repetition of eigenvalues), $t = 1, \dots, p$, and their corresponding eigenvectors $d^t = (d_1^t, \dots, d_p^t)^{\mathrm{T}}$ are orthogonal in $\mathbb{R}^p$, that is $\sum_{k=1}^p d_k^t d_k^j = \delta_{t,j}$ for $t, j = 1, \dots, p$. Then, μ^ε splits into p branches from μ^0,

$$\mu^\varepsilon = \mu^0 + \delta_\varepsilon \mu_t^1 + \mathrm{o}(\delta_\varepsilon), \quad t = 1, \dots, p,$$

with the corresponding eigenfunctions

$$\begin{aligned} \mathbf{U}^\varepsilon &= \delta_\varepsilon^{1/2}\big(d_1^t V_{1/2}^1 + \cdots + d_p^t V_{1/2}^p\big) + \mathrm{o}\big(\delta_\varepsilon^{1/2}\big) \quad \text{in } \Omega,\ t = 1, \dots, p, \\ \mathbf{w}^\varepsilon &= \big(d_1^t w^1 + \cdots + d_p^t w^p\big) + \mathrm{o}(1) \quad \text{on } \Gamma,\ t = 1, \dots, p. \end{aligned}$$

Following the process, we can determine all terms of the expansions.

Resonance case, $\boldsymbol{\mu^0 \in \sigma(P_\Omega) \cap \sigma(P_\Gamma)}$: *then, we prove that the asymptotic series* (3.1)–(3.3) *split into* $\kappa_\Omega(\mu^0) + \kappa_\Gamma(\mu^0)$ *branches.*

We consider μ^0 an eigenvalue of (2.15) ((2.16) respectively) and denote its multiplicity $\kappa_\Omega(\mu^0) = q$ ($\kappa_\Gamma(\mu^0) = p$, with $p = 1$ or $p = 2$, respectively). Let $V^1, \dots, V^q$ be the eigenfunctions of (2.15) associated with μ^0, which are orthonormal in $L^2(\Omega)$, and let $w^1, \dots, w^p$ be the corresponding eigenfunctions of (2.16) associated with μ^0, which are orthonormal in $L_h^2(\Gamma)$. Then, V_0 and w_0 can be written as (3.5) and (3.14) respectively where α_i and d_i are certain constants such that $\sum_{i=1}^q |\alpha_i|^2 \geqslant 0$ and $\sum_{j=1}^p |d_j|^2 \geqslant 0$ to be determined in the following steps.

In a second step we have that $V_{1/2}$ and $w_{1/2}$ satisfy problems (3.6) and (3.7) respectively. Now, the compatibility conditions for both non-homogeneous problems read:

$$\sum_{k=1}^p d_k \int_\Gamma \partial_\nu V^i w^k \,\mathrm{d}\tau = \mu^{1/2}\alpha_i \quad \text{for } i = 1, \dots, q,$$

and

$$\sum_{k=1}^q \alpha_k \int_\Gamma \partial_\nu V^k w^j \,\mathrm{d}\tau = \mu^{1/2} d_j \quad \text{for } j = 1, \dots, p.$$

That is, the vectors $\alpha = (\alpha_1, \dots, \alpha_q)^{\mathrm{T}}$ and $d = (d_1, \dots, d_p)^{\mathrm{T}}$ verify $\mathbf{A}d = \mu^{1/2}\alpha$ and $\mathbf{A}^{\mathrm{T}}\alpha = \mu^{1/2} d$ where $\mathbf{A}$ is the $q \times p$ matrix:

$$\mathbf{A} = \left(\int_\Gamma \partial_\nu V^i w^j \,\mathrm{d}\tau \right)_{i=1,\dots,q,\ j=1,\dots,p}. \tag{3.18}$$

Let n denote the rank of this matrix, $n \leqslant q$ and $n \leqslant p \leqslant 2$. Depending on whether $\mu^{1/2}$ is zero or not, we have different behavior.

First, we assume that $\mu^{1/2} \neq 0$, which amounts to $n > 0$. In this case, since α or d are different from zero, both vectors α and d are different from zero and, multiplying $\mathbf{A}^{\mathrm{T}}\alpha = \mu^{1/2} d$ by $\mu^{1/2}$ we have $\mathbf{A}^{\mathrm{T}}\mathbf{A}d = (\mu^{1/2})^2 d$. Then, $(\mu^{1/2})^2$ is an eigenvalue of the matrix $\mathbf{A}^{\mathrm{T}}\mathbf{A}$ with d the corresponding eigenvector and $\alpha = (\mu^{1/2})^{-1}\mathbf{A}d$.

Now we can check that $\mathbf{A}^{\mathrm{T}}\mathbf{A}$ is a $p \times p$ symmetric, semi-definite positive matrix, of rank n, and consequently, $\mathbf{A}^{\mathrm{T}}\mathbf{A}$ has n strictly positive eigenvalues μ_r (with the usual convention of repetition of eigenvalues), $r = 1, \dots, n$. Their corresponding eigenvectors $d^r = (d_1^r, \dots, d_p^r)^{\mathrm{T}}$ are orthogonal in $\mathbb{R}^p$. Therefore, we have that μ^ε splits from μ^0 into $2n$ branches with $\mu^{1/2} \neq 0$:

$$\mu^\varepsilon = \mu^0 \pm \delta_\varepsilon^{1/2}\sqrt{\mu_r} + \mathrm{o}\big(\delta_\varepsilon^{1/2}\big), \quad r = 1, \dots, n,$$

the corresponding eigenfunctions being,

$$\begin{aligned} \mathbf{U}^\varepsilon &= \pm\big(\alpha_1^r V^1 + \cdots + \alpha_q^r V^q\big) + \mathrm{o}(1) \quad \text{in } \Omega,\ r = 1, \dots, n, \\ \mathbf{w}^\varepsilon &= \big(d_1^r w^1 + \cdots + d_p^r w^p\big) + \mathrm{o}(1) \quad \text{on } \Gamma,\ r = 1, \dots, n, \end{aligned}$$

where $\alpha^r = (\sqrt{\mu_r})^{-1}\mathbf{A}d^r$. We can continue the process and determine all terms of the expansions.

In order to obtain the $\kappa_\Omega(\mu^0) + \kappa_\Gamma(\mu^0) = q + p$ different branches we assume that $\mu^{1/2} = 0$. Now $\mathbf{A}d = \mathbf{A}^{\mathrm{T}}\alpha = 0$ and there exist $p - n$ vectors $d^t \in \mathbb{R}^p$ and $q - n$ vectors $\alpha^s \in \mathbb{R}^q$ such that

$$\mathbf{A}d^t = 0 \quad \text{and} \quad \sum_{k=1}^{p} d_k^t d_k^j = \delta_{t,j} \quad \text{for } t, j = 1, \ldots, p - n, \quad \text{and} \tag{3.19}$$

$$\mathbf{A}^{\mathrm{T}}\alpha^s = 0 \quad \text{and} \quad \sum_{k=1}^{q} \alpha_k^s \alpha_k^i = \delta_{s,i} \quad \text{for } s, i = 1, \ldots, q - n. \tag{3.20}$$

For each $t = 1, \ldots, p - n$ and $s = 1, \ldots, q - n$, let us define $V_{1/2}^t$ and $w_{1/2}^s$ as the solutions of:

$$\begin{cases} -\Delta_x V_{1/2}^t = \mu^0 V_{1/2}^t & \text{in } \Omega, \\ V_{1/2}^t = d_1^t w^1 + \cdots + d_p^t w^p & \text{on } \Gamma, \end{cases} \tag{3.21}$$

$$\partial_\tau\left(h\partial_\tau w_{1/2}^s\right) + \mu^0 h w_{1/2}^s = \partial_\nu\left(\alpha_1^s V^1 + \cdots + \alpha_q^s V^q\right) \quad \text{on } \Gamma, \tag{3.22}$$

respectively such that $\int_\Omega V_{1/2}^t V^i \,\mathrm{d}x = 0$ for $i = 1, \ldots, q$ and $\int_\Gamma h w_{1/2}^s w^j \,\mathrm{d}\tau = 0$ for $j = 1, \ldots, p$. These solutions exist on account of (3.19) and (3.20). Let us note that, for each $t = 1, \ldots, p - n$ and $s = 1, \ldots, q - n$, the functions $V_{1/2}^t$ and $w_{1/2}^s$ are well defined because of the construction of d^t and α^s. Then, the functions $V_{1/2}$ and $w_{1/2}$, satisfying problems (3.6) and (3.7) respectively with $\mu^{1/2} = 0$, can be written as follow:

$$V_{1/2} = \mathsf{d}_1 V_{1/2}^1 + \cdots + \mathsf{d}_{p-n} V_{1/2}^{p-n} + \beta_1 V^1 + \cdots + \beta_q V^q, \quad \text{and}$$
$$w_{1/2} = \mathsf{a}_1 w_{1/2}^1 + \cdots + \mathsf{a}_{q-n} w_{1/2}^{q-n} + f_1 w^1 + \cdots + f_p w^p,$$

where

$$\mathsf{d}_1 d_j^1 + \cdots + \mathsf{d}_{p-n} d_j^{p-n} = d_j \quad \text{for } j = 1, \ldots, p, \tag{3.23}$$

$$\mathsf{a}_1 \alpha_i^1 + \cdots + \mathsf{a}_{q-n} \alpha_i^{q-n} = \alpha_i \quad \text{for } i = 1, \ldots, q, \tag{3.24}$$

and β_i and f_j are certain constants to be determined below.

In the third step, we obtain problems (3.10) and (3.11). In this case, the compatibility conditions for both non-homogeneous problems read:

$$\mathsf{a}_1 \int_\Gamma \partial_\nu V^i w_{1/2}^1 \,\mathrm{d}\tau + \cdots + \mathsf{a}_{q-n} \int_\Gamma \partial_\nu V^i w_{1/2}^{q-n} \,\mathrm{d}\tau + \sum_{k=1}^{p} f_k \int_\Gamma \partial_\nu V^i w^k \,\mathrm{d}\tau = \mu^1 \alpha_i \quad \text{for } i = 1, \ldots, q, \tag{3.25}$$

$$\mathsf{d}_1 \int_\Gamma \partial_\nu V_{1/2}^1 w^j \,\mathrm{d}\tau + \cdots + \mathsf{d}_{p-n} \int_\Gamma \partial_\nu V_{1/2}^{p-n} w^j \,\mathrm{d}\tau + \sum_{k=1}^{q} \beta_k \int_\Gamma \partial_\nu V^k w^j \,\mathrm{d}\tau = \mu^1 d_j \quad \text{for } j = 1, \ldots, p. \tag{3.26}$$

For each s fixed, $s = 1, \ldots, q - n$, adding, from i equal 1 to q, Eqs. (3.25) once we have multiplied each one by α_i^s respectively, and using (3.20) and (3.24), we obtain:

$$\mathsf{a}_1 \sum_{i=1}^{q} \alpha_i^s \int_\Gamma \partial_\nu V^i w_{1/2}^1 \,\mathrm{d}\tau + \cdots + \mathsf{a}_{q-n} \sum_{i=1}^{q} \alpha_i^s \int_\Gamma \partial_\nu V^i w_{1/2}^{q-n} \,\mathrm{d}\tau = \mu^1 \mathsf{a}_s,$$

and μ^1 is an eigenvalue of the matrix,

$$\mathbf{B} = \left(\sum_{i=1}^{q} \alpha_i^s \int_\Gamma \partial_\nu V^i w_{1/2}^k \,\mathrm{d}\tau\right)_{s,k=1,\ldots,q-n}, \tag{3.27}$$

which is symmetric because of (3.22), and $\mathsf{a} = (\mathsf{a}_1, \ldots, \mathsf{a}_{q-n})^{\mathrm{T}}$ is the corresponding eigenvector. Once we have fixed the eigenvalue μ^1 and the eigenvector a, we determine α and f by (3.24) and (3.25). This gives us $q - n$ values μ^1 with their respective α.

Similarly, for each t fixed, $t=1,\dots,p-n$, adding, from j equal 1 to p, Eqs. (3.26) once we have multiplied each one by d_j^t respectively, and using (3.19) and (3.23), we get:

$$\mathsf{d}_1\sum_{j=1}^{p} d_j^t\int_{\Gamma}\partial_\nu V_{1/2}^1 w^j\,\mathrm{d}\tau+\cdots+\mathsf{d}_{p-n}\sum_{j=1}^{p} d_j^t\int_{\Gamma}\partial_\nu V_{1/2}^{p-n} w^j\,\mathrm{d}\tau=\mu^1\mathsf{d}_t,$$

and μ^1 is an eigenvalue of the matrix

$$\mathbf{C}=\left(\sum_{j=1}^{p} d_j^t\int_{\Gamma}\partial_\nu V_{1/2}^k w^j\,\mathrm{d}\tau\right)_{t,k=1,\dots,p-n},\tag{3.28}$$

which is symmetric on account of (3.21), and $\mathsf{d}=(\mathsf{d}_1,\dots,\mathsf{d}_{p-n})^{\mathrm{T}}$ is the corresponding eigenvector. Once we have fixed the eigenvalue μ^1 and the eigenvector d, we determine d and β by (3.23) and (3.26). This gives us $p-n$ values μ^1 with their respective d.

Therefore, we have shown that μ^ε splits from μ^0 into $p+q-2n$ branches with $\mu^{1/2}=0$:

$$\mu^\varepsilon=\mu^0+\delta_\varepsilon\mu^1+\mathrm{o}(\delta_\varepsilon),$$

μ^1 being an eigenvalue of $\mathbf{B}$ or $\mathbf{C}$, with the corresponding eigenfunctions:

$$\mathbf{U}^\varepsilon=\left(\alpha_1V^1+\cdots+\alpha_qV^q\right)+\mathrm{o}(1)\quad\text{in }\Omega,$$
$$\mathbf{w}^\varepsilon=\left(d_1w^1+\cdots+d_pw^p\right)+\mathrm{o}(1)\quad\text{on }\Gamma,$$

where α and d are determined above. Let us note that α and d may be different from zero simultaneously if μ^1 is an eigenvalue of $\mathbf{B}$ and $\mathbf{C}$ while α (d respectively) is zero if μ^1 is an eigenvalue of $\mathbf{C}$ ($\mathbf{B}$ respectively) but not of $\mathbf{B}$ ($\mathbf{C}$ respectively). Again, following the process, laborious computations allow the other terms of the asymptotic expansions to be determined.

Remark 3.1. We observe that in any case the method used throughout this section shows that μ^ε splits into $\kappa_\Omega(\mu^0)+\kappa_\Gamma(\mu^0)$ branches from the unique eigenvalue μ^0 of problem (2.15) or of problem (2.16). We also observe that the power $\delta_\varepsilon^{1/2}$ in (3.1) can only appear in the case where $\mu^0\in\sigma(P_\Omega)\cap\sigma(P_\Gamma)$ and $\mathrm{rank}(\mathbf{A})>0$ (see Proposition 5.1 to compare).

4. Convergence for the eigenelements of problem (1.1)

In this section, we justify the asymptotic expansions in Section 3 up to a certain degree. First, we prove the convergence as $\varepsilon\to0$ of the eigenvalues and eigenfunctions of problem (2.4) towards those of problem (2.15)–(2.16), in the space $H^1(\Omega)\times H^1(\Gamma)$, with conservation of multiplicity as stated in Theorems 4.1–4.2. Then, we prove that the rest of the terms in (3.1)–(3.3) provide true correcting terms for the eigenelements of (2.4), improving the convergence of the eigenelements towards the leading term $\{\mu^0,V_0,w_0\}$ in (3.1)–(3.3) as asserted by Theorems 4.3–4.7. Obviously, these correcting terms depend on whether $\mu^0\in\sigma(P_\Omega)\cap\sigma(P_\Gamma)$ or exclusively $\mu^0\in\sigma(P_\Omega)$ or $\mu^0\in\sigma(P_\Gamma)$ (see Remark 4.2). Precise bounds for convergence rates of the eigenelements are also provided, which in fact depend on ε and the eigenvalue number k. These bounds can be improved by specifying the precise dependence on both parameters and on properties of the limiting spectrum as we outline in Theorem A.1 (see Remark 4.1).

For this section we recall the normalization (2.8) for the eigenelements of (2.7) and introduce a continuous extension operator from $H^1(\Gamma)$ into $H^1(\Omega)$ as follows:

Let φ be a fixed function such that $\varphi\in C^\infty(\mathbb{R})$, $0\leqslant\varphi\leqslant1$, $\varphi(r)=0$ if $r\leqslant1$ and $\varphi(r)=1$ if $r\geqslant2$. For each $w\in H^1(\Gamma)$ we define:

$$U_w(x)=w(\tau)\varphi(2+\nu/d)\quad\text{for }x\in\Omega,\tag{4.1}$$

which satisfies,

$$U_w\in H^1(\Omega),\quad U_w=w\quad\text{on }\Gamma\quad\text{and}\quad\|U_w\|_{H^1(\Omega)}\leqslant C\|w\|_{H^1(\Gamma)},\tag{4.2}$$

where C is a constant independent of ε and w.

Theorem 4.1. *The eigenvalues μ_k^ε of* (2.7) *converge towards the eigenvalues μ_k^0 of* (2.17) *as $\varepsilon \to 0$, and there is conservation of the multiplicity. More specifically, for each fixed k, $k = 0, 1, 2 \ldots$, there exist constants C_k and $\varepsilon_k > 0$ such that, for $\varepsilon \leqslant \varepsilon_k$,*

$$\left|\mu_k^\varepsilon - \mu_k^0\right| \leqslant C_k \delta_\varepsilon^{1/2}. \tag{4.3}$$

Moreover, for any eigenvalue μ_k^0 of (2.17) *with multiplicity $\varkappa_k$ ($\mu_k^0 = \mu_{k+1}^0 = \cdots = \mu_{k+\varkappa_k-1}^0$), and for any eigenfunction (V, w) of* (2.17) *associated with μ_k^0 with $\|\!|(V, w)|\!\| = 1$, there is a linear combination $(\widetilde{\mathbf{U}}^\varepsilon, \tilde{\mathbf{w}}^\varepsilon)$ of eigenfunctions associated with $\{\mu_i^\varepsilon\}_{i=k}^{k+\varkappa_k-1}$ such that*

$$\left|\!\left|\!\left|\left(\widetilde{\mathbf{U}}^\varepsilon, \tilde{\mathbf{w}}^\varepsilon\right) - (V, w)\right|\!\right|\!\right| \leqslant M_k \delta_\varepsilon^{1/2}, \tag{4.4}$$

where the constant M_k is independent of ε.

In addition, for each sequence $(\mathbf{U}_k^\varepsilon, \mathbf{w}_k^\varepsilon)$ of eigenfunctions of (2.7), *$\|\!|(\mathbf{U}_k^\varepsilon, \mathbf{w}_k^\varepsilon)|\!\| = 1$, we can extract a subsequence (still denoted by ε) such that $(\mathbf{U}_k^\varepsilon, \mathbf{w}_k^\varepsilon) \to (V_k^*, w_k^*)$ in $H^1(\Omega) \times H^1(\Gamma)$, as $\varepsilon \to 0$, where (V_k^*, w_k^*) is an eigenfunction of* (2.17) *associated with μ_k^0 and the set $\{(V_k^*, w_k^*)\}_{k=0}^\infty$ forms an orthonormal basis of $H_0^1(\Omega) \times H^1(\Gamma)$ for the scalar product* (2.6).

Proof. Let us consider the Hilbert space $\mathcal{H}_\varepsilon$ defined by (2.5) with the scalar product (2.6). Let A_ε be the positive, selfadjoint and compact operator defined on $\mathcal{H}_\varepsilon$ as follows: for any $(F, f) \in \mathcal{H}_\varepsilon$, $A_\varepsilon(F, f) = (U^\varepsilon, u^\varepsilon)$ where $(U^\varepsilon, u^\varepsilon) \in \mathcal{H}_\varepsilon$ is the unique solution of

$$\left(\!\left(\left(U^\varepsilon, u^\varepsilon\right), (G, g)\right)\!\right) = \int_\Omega F G \,\mathrm{d}x + \int_\Gamma h f g \,\mathrm{d}\tau \quad \forall (G, g) \in \mathcal{H}_\varepsilon. \tag{4.5}$$

Obviously, the eigenvalues of A_ε are $\{(1 + \mu_k^\varepsilon)^{-1}\}_{k=0}^\infty$ where $\{\mu_k^\varepsilon\}_{k=0}^\infty$ are the eigenvalues of (2.7).

In the same way, we consider the Hilbert space $\mathcal{H}_0 = H_0^1(\Omega) \times H^1(\Gamma)$ with the scalar product (2.6) and define the operator A_0 on $\mathcal{H}_0$ by $A_0(F, f) = (U, u)$, for $(F, f) \in \mathcal{H}_0$, where (U, u) is the unique solution of

$$\left(\!\left((U, u), (G, g)\right)\!\right) = \int_\Omega F G \,\mathrm{d}x + \int_\Gamma h f g \,\mathrm{d}\tau \quad \forall (G, g) \in \mathcal{H}_0. \tag{4.6}$$

The eigenvalues of A_0 are $\{(1 + \mu_k^0)^{-1}\}_{k=0}^\infty$ where $\{\mu_k^0\}_{k=0}^\infty$ are the eigenvalues of (2.17).

Let R_ε be the linear, continuous operator $R_\varepsilon : \mathcal{H}_0 \to \mathcal{H}_\varepsilon$ defined by $R_\varepsilon(F, f) = (\widehat{F}^\varepsilon, f)$ for any $(F, f) \in H_0^1(\Omega) \times H^1(\Gamma)$, where

$$\widehat{F}^\varepsilon(x) = F(x) + \delta_\varepsilon^{1/2} U_f(x) \quad \text{for } x \in \Omega, \tag{4.7}$$

with U_f the function defined by (4.1). Let $\mathcal{W}$ be the space $\mathcal{H}_0$.

In order to apply Lemma 2.3 we check the properties (a)–(d) in this lemma.

On account of (4.7) and (4.2), for any $(F, f) \in H^1(\Omega) \times H^1(\Gamma)$,

$$\left\|\widehat{F}^\varepsilon - F\right\|_{H^1(\Omega)} \leqslant C \delta_\varepsilon^{1/2} \|f\|_{H^1(\Gamma)}. \tag{4.8}$$

Also, for $(F, f) \in H_0^1(\Omega) \times H^1(\Gamma)$, we have the convergence,

$$R_\varepsilon(F, f) \to (F, f) \quad \text{in } H_0^1(\Omega) \times H^1(\Gamma), \quad \text{as } \varepsilon \to 0. \tag{4.9}$$

Consequently, $\|\!|R_\varepsilon(F, f)|\!\| = \|\!|(\widehat{F}^\varepsilon, f)|\!\| \to \|\!|(F, f)|\!\|$ when $\varepsilon \to 0$, and property (a) is satisfied.

In order to prove the uniform bound for $\|A_\varepsilon\|_{\mathcal{L}(\mathcal{H}_\varepsilon)}$, for any $(F, f) \in \mathcal{H}_\varepsilon$, we consider $A_\varepsilon(F, f) = (U^\varepsilon, u^\varepsilon)$ to be the solution of (4.5) and take $(G, g) = (U^\varepsilon, u^\varepsilon)$. Applying the Cauchy–Buniakowsky–Schwarz inequality, we obtain:

$$\left|\!\left|\!\left|\left(U^\varepsilon, u^\varepsilon\right)\right|\!\right|\!\right| \leqslant C\left(\|F\|_{L^2(\Omega)} + \|f\|_{L_h^2(\Gamma)}\right).$$

Then, property (b) is satisfied, on account of

$$\|A_\varepsilon\|_{\mathcal{L}(\mathcal{H}_\varepsilon)} = \sup_{(F,f) \in \mathcal{H}_\varepsilon, (F,f) \neq 0} \frac{\|\!|A_\varepsilon(F, f)|\!\|}{\|\!|(F, f)|\!\|} \leqslant C.$$

Let us prove property (c). For each $\varepsilon > 0$ and any fixed $(F, f) \in \mathcal{H}_0$, we consider $(U^\varepsilon, u^\varepsilon) = A_\varepsilon R_\varepsilon(F, f)$; $(U^\varepsilon, u^\varepsilon) \in \mathcal{H}_\varepsilon$ satisfies (4.5) for $(F, f) = (\widehat{F}^\varepsilon, f)$, that is,

$$\left(\left(\left(U^\varepsilon, u^\varepsilon\right), (G, g)\right)\right) = \int_\Omega \widehat{F}^\varepsilon G \,\mathrm{d}x + \int_\Gamma hfg \,\mathrm{d}\tau \quad \forall (G, g) \in \mathcal{H}_\varepsilon. \tag{4.10}$$

On account of (4.8), we consider (4.10) with $(G, g) = (U^\varepsilon, u^\varepsilon)$ and we obtain that, for ε sufficiently small, $|||(U^\varepsilon, u^\varepsilon)|||$ is bounded by a constant independent of ε. Therefore, we can extract a subsequence, still denoted by ε, such that $(U^\varepsilon, u^\varepsilon)$ converges, as $\varepsilon \to 0$, towards some function (U^*, u^*) weakly in $H^1(\Omega) \times H^1(\Gamma)$. Since $(U^\varepsilon, u^\varepsilon) \in \mathcal{H}_\varepsilon$, $U^\varepsilon = \delta_\varepsilon^{1/2} u^\varepsilon$ on Γ and $U^* = 0$ on Γ. In order to identify $(U^*, u^*) \in \mathcal{H}_0$ with $A_0(F, f)$, for any $(V, v) \in \mathcal{H}_0$ fixed, we consider (4.10) for $(G, g) = R_\varepsilon(V, v)$ and pass to the limit when $\varepsilon \to 0$. Then, by virtue of (4.9) and the convergence of $(U^\varepsilon, u^\varepsilon)$ we obtain that (U^*, u^*) satisfies (4.6) and, in consequence, $(U^*, u^*) = A_0(F, f)$. In addition, taking limits in (4.10) for $(G, g) = (U^\varepsilon, u^\varepsilon)$, we have:

$$\left|\left|\left|\left(U^\varepsilon, u^\varepsilon\right)\right|\right|\right|^2 = \int_\Omega \widehat{F}^\varepsilon U^\varepsilon \,\mathrm{d}x + \int_\Gamma hfu^\varepsilon \,\mathrm{d}\tau \xrightarrow{\varepsilon \to 0} \int_\Omega FU^* \,\mathrm{d}x + \int_\Gamma hfu^* \,\mathrm{d}\tau = \left|\left|\left|\left(U^*, u^*\right)\right|\right|\right|^2,$$

and $(U^\varepsilon, u^\varepsilon)$ converge towards (U^*, u^*) strongly in $H^1(\Omega) \times H^1(\Gamma)$. Thus, because of (4.9),

$$\left|\left|\left|A_\varepsilon R_\varepsilon(F, f) - R_\varepsilon A_0(F, f)\right|\right|\right| = \left|\left|\left|\left(U^\varepsilon, u^\varepsilon\right) - R_\varepsilon\left(U^*, u^*\right)\right|\right|\right| \xrightarrow{\varepsilon \to 0} 0 \quad \text{for any } (F, f) \in \mathcal{H}_0$$

and property (c) of Lemma 2.3 is proved.

In a similar way to property (c), we can prove that for any sequence $(F^\varepsilon, f^\varepsilon)$ in $\mathcal{H}_\varepsilon$ such that $\sup_\varepsilon |||(F^\varepsilon, f^\varepsilon)|||$ is bounded by a constant independent of ε, we can extract a subsequence, still denoted by ε, verifying $|||A_\varepsilon(F^\varepsilon, f^\varepsilon) - R_\varepsilon(V_0, v_0)||| \to 0$ as $\varepsilon \to 0$, for a certain function $(V_0, v_0) \in \mathcal{H}_0$. Thus, A_ε is uniformly compact, and property (d) of Lemma 2.3 holds.

Now, Lemma 2.3 leads us to assert that for each fixed k, there exist constants C_k and $\varepsilon_k > 0$ such that for $\varepsilon \leqslant \varepsilon_k$,

$$\left|\left(1 + \mu_k^\varepsilon\right)^{-1} - \left(1 + \mu_k^0\right)^{-1}\right| \leqslant C_k \sup \left|\left|\left|A_\varepsilon R_\varepsilon(V, w) - R_\varepsilon A_0(V, w)\right|\right|\right|,$$

where the sup is taken over all (V, w) such that $|||(V, w)||| = 1$, (V, w) in the eigenspace associated with μ_k^0. Moreover, for any eigenvalue μ_k^0 of (2.17) with multiplicity $\varkappa_k$ ($\mu_k^0 = \mu_{k+1}^0 = \cdots = \mu_{k+\varkappa_k-1}^0$), and for any eigenfunction (V, w) of (2.17) associated with μ_k^0 with $|||(V, w)||| = 1$, there is a linear combination $(\widetilde{\mathbf{U}}^\varepsilon, \tilde{\mathbf{w}}^\varepsilon)$ of eigenfunctions associated with $\{\mu_i^\varepsilon\}_{i=k}^{k+\varkappa_k-1}$ such that

$$\left|\left|\left|\left(\widetilde{\mathbf{U}}^\varepsilon, \tilde{\mathbf{w}}^\varepsilon\right) - R_\varepsilon(V, w)\right|\right|\right| \leqslant M_k \left|\left|\left|A_\varepsilon R_\varepsilon(V, w) - R_\varepsilon A_0(V, w)\right|\right|\right|,$$

where the constant M_k is independent of ε. Thus, from (4.8), estimates (4.3) and (4.4) hold once we prove that for any eigenfunction (V, w) of (2.17) associated with μ_k^0 of norm $|||(V, w)||| = 1$,

$$\left|\left|\left|A_\varepsilon R_\varepsilon(V, w) - R_\varepsilon A_0(V, w)\right|\right|\right| \leqslant C\delta_\varepsilon^{1/2}, \tag{4.11}$$

for C a constant independent of C and (V, w).

Indeed, let us denote by $(U^\varepsilon, u^\varepsilon)$ and (U, u) the functions $A_\varepsilon R_\varepsilon(V, w)$ and $A_0(V, w)$ respectively. Then, $(U^\varepsilon, u^\varepsilon) \in \mathcal{H}_\varepsilon$ satisfies (4.5) for $(F, f) = (\widehat{V}^\varepsilon, w)$, that is,

$$\left(\left(\left(U^\varepsilon, u^\varepsilon\right), (G, g)\right)\right) = \int_\Omega \widehat{V}^\varepsilon G \,\mathrm{d}x + \int_\Gamma hwg \,\mathrm{d}\tau \quad \forall (G, g) \in \mathcal{H}_\varepsilon. \tag{4.12}$$

Taking $(G, g) = (U^\varepsilon, u^\varepsilon)$ in (4.12) and using (4.9) and the fact that $|||(V, w)||| = 1$, we have that $|||(U^\varepsilon, u^\varepsilon)|||$ is bounded for ε sufficiently small. Moreover, taking $(G, g) = (U^\varepsilon - U - \delta_\varepsilon^{1/2} U_u, u^\varepsilon - u) \in \mathcal{H}_\varepsilon$ in (4.12) where U_u is the function defined by (4.1) we obtain:

$$\left(\left(\left(U^\varepsilon, u^\varepsilon\right), \left(U^\varepsilon - U - \delta_\varepsilon^{1/2} U_u, u^\varepsilon - u\right)\right)\right) = \int_\Omega \widehat{V}^\varepsilon \left(U^\varepsilon - U - \delta_\varepsilon^{1/2} U_u\right) \mathrm{d}x + \int_\Gamma hw\left(u^\varepsilon - u\right) \mathrm{d}\tau. \tag{4.13}$$

On the other hand, we can prove that $(U,u)\in\mathcal{H}_0$ verifies $|||(U,u)|||\leqslant 1$, and

$$\Big(\big((U,u),\big(U^\varepsilon-\delta_\varepsilon^{1/2}U_{u^\varepsilon}-U,u^\varepsilon-u\big)\big)\Big)=\int_\Omega V\big(U^\varepsilon-\delta_\varepsilon^{1/2}U_{u^\varepsilon}-U\big)\,\mathrm{d}x+\int_\Gamma hw\big(u^\varepsilon-u\big)\,\mathrm{d}\tau, \tag{4.14}$$

where $(U^\varepsilon-\delta_\varepsilon^{1/2}U_{u^\varepsilon}-U,u^\varepsilon-u)\in\mathcal{H}_0$. Thus, combining (4.13) and (4.14) yields:

$$\big|\big|\big|\big(U^\varepsilon-U,u^\varepsilon-u\big)\big|\big|\big|^2=I_1+I_2+I_3+I_4,$$

where

$$I_1=\int_\Omega\big(\widehat{V}^\varepsilon-V\big)\big(U^\varepsilon-U\big)\,\mathrm{d}x,$$

$$I_2=-\delta_\varepsilon^{1/2}\int_\Omega\widehat{V}^\varepsilon U_u\,\mathrm{d}x+\delta_\varepsilon^{1/2}\int_\Omega VU_{u^\varepsilon}\,\mathrm{d}x,$$

$$I_3=\delta_\varepsilon^{1/2}\int_\Omega\nabla_xU^\varepsilon\cdot\nabla_xU_u\,\mathrm{d}x-\delta_\varepsilon^{1/2}\int_\Omega\nabla_xU\cdot\nabla_xU_{u^\varepsilon}\,\mathrm{d}x,$$

and

$$I_4=\delta_\varepsilon^{1/2}\int_\Omega U^\varepsilon U_u\,\mathrm{d}x-\delta_\varepsilon^{1/2}\int_\Omega UU_{u^\varepsilon}\,\mathrm{d}x.$$

On account of the Cauchy–Buniakowsky–Schwarz inequality, (4.8), (2.21), (4.2) and the bounds $|||(V,w)|||=1$ and $|||(U,u)|||\leqslant 1$, we have, for $\varepsilon\leqslant\varepsilon_0$,

$$|I_1|\leqslant\big\|\widehat{V}^\varepsilon-V\big\|_{L^2(\Omega)}\big\|U^\varepsilon-U\big\|_{L^2(\Omega)}\leqslant C\delta_\varepsilon^{1/2}\big|\big|\big|\big(U^\varepsilon-U,u^\varepsilon-u\big)\big|\big|\big|,$$

$$\begin{aligned}|I_2|&=\delta_\varepsilon^{1/2}\bigg|\int_\Omega VU_{u^\varepsilon-u}\,\mathrm{d}x+\int_\Omega\big(V-\widehat{V}^\varepsilon\big)U_u\,dx\bigg|\\&\leqslant C\delta_\varepsilon^{1/2}\big(\|V\|_{L^2(\Omega)}\big\|u^\varepsilon-u\big\|_{L^2(\Gamma)}+\delta_\varepsilon^{1/2}\|u\|_{L^2(\Gamma)}\big)\leqslant C\delta_\varepsilon^{1/2}\big(\big|\big|\big|\big(U^\varepsilon-U,u^\varepsilon-u\big)\big|\big|\big|+\delta_\varepsilon^{1/2}\big),\\|I_3|&=\delta_\varepsilon^{1/2}\bigg|\int_\Omega\nabla_x\big(U^\varepsilon-U\big)\cdot\nabla_xU_u\,\mathrm{d}x+\int_\Omega\nabla_xU\cdot\nabla_xU_{u-u^\varepsilon}\,\mathrm{d}x\bigg|\\&\leqslant C\delta_\varepsilon^{1/2}\big(\big\|\nabla_x\big(U^\varepsilon-U\big)\big\|_{L^2(\Omega)}\|u\|_{H^1(\Gamma)}+\|\nabla U\|_{L^2(\Omega)}\big\|u^\varepsilon-u\big\|_{H^1(\Gamma)}\big)\leqslant C\delta_\varepsilon^{1/2}\big|\big|\big|\big(U^\varepsilon-U,u^\varepsilon-u\big)\big|\big|\big|,\\|I_4|&=\delta_\varepsilon^{1/2}\bigg|\int_\Omega\big(U^\varepsilon-U\big)U_u\,\mathrm{d}x+\int_\Omega UU_{u-u^\varepsilon}\,\mathrm{d}x\bigg|\\&\leqslant C\delta_\varepsilon^{1/2}\big(\big\|U^\varepsilon-U\big\|_{L^2(\Omega)}\|u\|_{L^2(\Gamma)}+\|U\|_{L^2(\Omega)}\big\|u^\varepsilon-u\big\|_{L^2(\Gamma)}\big)\leqslant C\delta_\varepsilon^{1/2}\big|\big|\big|\big(U^\varepsilon-U,u^\varepsilon-u\big)\big|\big|\big|.\end{aligned}$$

Then, $|||(U^\varepsilon-U,u^\varepsilon-u)|||^2\leqslant C(\delta_\varepsilon+\delta_\varepsilon^{1/2}|||(U^\varepsilon-U,u^\varepsilon-u)|||)$ with C a constant independent of ε and hence the inequality $|||A_\varepsilon R_\varepsilon(V,w)-A_0(V,w)|||=|||(U^\varepsilon-U,u^\varepsilon-u)|||\leqslant C\delta_\varepsilon^{1/2}$ holds. Finally, (4.8) leads us to (4.11), which concludes the proof of (4.3) and (4.4).

As regards the proof of the last statement in the theorem, we consider the sequence $(\mathbf{U}_k^\varepsilon,\mathbf{w}_k^\varepsilon)$ of eigenfunctions of (2.7), $|||(\mathbf{U}_k^\varepsilon,\mathbf{w}_k^\varepsilon)|||=1$. By a classical argument of diagonalization we extract a subsequence (still denoted by ε) such that $(\mathbf{U}_k^\varepsilon,\mathbf{w}_k^\varepsilon)\to(V_k^*,w_k^*)$ in $H^1(\Omega)\times H^1(\Gamma)$-weak, as $\varepsilon\to 0$. If we assume that this limit $(V_k^*,w_k^*)\neq 0$, on account of (4.8) and the fact that $\mu_k^\varepsilon\to\mu_k^0$ as $\varepsilon\to 0$, by taking limit in (2.7) for $(W,w)=R_\varepsilon(V,v)$ and any fixed $(V,v)\in H_0^1(\Omega)$, we identify (V_k^*,w_k^*) with an eigenfunction of (2.17) associated with μ_k^0. Then, using again the convergence $\mu_k^\varepsilon\to\mu_k^0$ and (2.7) for $(W,w)=(\mathbf{U}_k^\varepsilon,\mathbf{w}_k^\varepsilon)$, we get $1=|||(\mathbf{U}_k^\varepsilon,\mathbf{w}_k^\varepsilon)|||^2\to(\mu_k^0+1)(\|U_k^*\|^2_{L^2(\Omega)}+\|w_k^*\|^2_{L^2_h(\Gamma)})$ and prove that $(V_k^*,w_k^*)\neq 0$ and the strong convergence of the sequence $(\mathbf{U}_k^\varepsilon,\mathbf{w}_k^\varepsilon)$ in $H^1(\Omega)\times H^1(\Gamma)$ towards (V_k^*,w_k^*). The fact that the (V_k^*,w_k^*) are orthogonal in $H^1(\Omega)\times H^1(\Gamma)$ for the scalar product (2.6) follows from the orthogonality condition for $(\mathbf{U}_k^\varepsilon, \mathbf{w}_k^\varepsilon)$. The fact that the set $\{(V_k^*, w_k^*)\}_{k=0}^\infty$ forms a basis of $H_0^1(\Omega) \times H^1(\Gamma)$ is obtained by contradiction since all the eigenvalues of (2.17) have finite multiplicity. Therefore, the theorem is proved. □

Now, using Lemma 2.2, in the following theorem we provide an alternative bound for the convergence rate of the eigenfunctions.

Theorem 4.2. *Let μ_k^0 be an eigenvalue of problem* (2.17) *with multiplicity $\varkappa_k$, $\mu_k^0 = \cdots = \mu_{k+\varkappa_k-1}^0$, and let $\{(V_j, w_j)\}_{j=k}^{k+\varkappa_k-1}$ be the associated eigenfunctions verifying $(((V_j, w_j), (V_i, w_i))) = \delta_{i,j}$ for $i, j = k, \ldots, k+\varkappa_k-1$. Let $\{(\mathbf{U}_q^\varepsilon, \mathbf{w}_q^\varepsilon)\}_{q=k}^{k+\varkappa_k-1}$ be the eigenfunctions corresponding to the eigenvalues $\mu_k^\varepsilon, \ldots, \mu_{k+\varkappa_k-1}^\varepsilon$ of problem* (2.7) *with $(((\mathbf{U}_j^\varepsilon, \mathbf{w}_j^\varepsilon), (\mathbf{U}_i^\varepsilon, \mathbf{w}_i^\varepsilon))) = \delta_{i,j}$ for $i, j = k, \ldots, k+\varkappa_k-1$. Then, for $\varepsilon < \varepsilon_k^*$ and $q = k, \ldots, k+\varkappa_k-1$ there exist coefficients $\beta_q^{(j)}(\varepsilon)$ such that*

$$\left|\!\left|\!\left|\left(\mathbf{U}_q^\varepsilon, \mathbf{w}_q^\varepsilon\right) - \sum_{q=j}^{k+\varkappa_k-1} \beta_q^{(j)}(\varepsilon)(V_j, w_j)\right|\!\right|\!\right| \leqslant \widetilde{M}_k \delta_\varepsilon^{1/2} \tag{4.15}$$

where $\widetilde{M}_k$ is a constant independent of ε.

Proof. In order to apply Lemma 2.2, we consider $H = \mathcal{H}_\varepsilon$ the Hilbert space defined by (2.5), $n = N = \varkappa_k$, $w^i = (\mathbf{U}_{k+i-1}^\varepsilon, \mathbf{w}_{k+i-1}^\varepsilon)$ and $W^j = |||\widetilde{W}^j|||^{-1}\widetilde{W}^j$, for $i, j = 1, \ldots, \varkappa_k$, where $\widetilde{W}^j = (V_{k+j-1} + \delta_\varepsilon^{1/2} U_{w_{k+j-1}}, w_{k+j-1}) \in \mathcal{H}_\varepsilon$ with U_w the function defined by (4.1). It is clear that hypothesis (a) of Lemma 2.2 holds true.

Let us prove property (b). By definition of $\widetilde{W}^j$, (2.17) and the normalization of (V_p, w_p), we have

$$\begin{aligned}\left(\!\left(\widetilde{W}_j, \widetilde{W}_i\right)\!\right) - \delta_{i,j} = {} & \delta_\varepsilon^{1/2} \int_\Omega \nabla_x V_{k+j-1} \cdot \nabla_x U_{w_{k+i-1}} \, \mathrm{d}x + \delta_\varepsilon^{1/2} \int_\Omega \nabla_x U_{w_{k+j-1}} \cdot \nabla_x V_{k+i-1} \, \mathrm{d}x \\ & + \delta_\varepsilon \int_\Omega \nabla_x U_{w_{k+j-1}} \cdot \nabla_x U_{w_{k+i-1}} \, \mathrm{d}x + \delta_\varepsilon^{1/2} \int_\Omega V_{k+j-1} U_{w_{k+i-1}} \, \mathrm{d}x \\ & + \delta_\varepsilon^{1/2} \int_\Omega U_{w_{k+j-1}} V_{k+i-1} \, \mathrm{d}x + \delta_\varepsilon \int_\Omega U_{w_{k+j-1}} U_{w_{k+i-1}} \, \mathrm{d}x,\end{aligned}$$

and $|((\widetilde{W}_j, \widetilde{W}_i)) - \delta_{i,j}| \leqslant C\delta_\varepsilon^{1/2}$. In particular, at $i = j$, $|\,|||\widetilde{W}^j|||^2 - 1| \leqslant C\delta_\varepsilon^{1/2}$ and consequently, for ε sufficiently small, $|||\widetilde{W}^j||| > C_0$ with C_0 a constant independent of ε. Thus, from (4.8), the hypothesis (b) holds for $\varrho = \widetilde{C}\delta_\varepsilon^{1/2}$ where $\widetilde{C}$ is a constant independent of ε.

Finally, on account of (4.4) and the fact the $|||W^j - (V_{k+j-1}, w_{k+j-1})|||^2 \leqslant C\delta_\varepsilon$, hypothesis (c) of Lemma 2.2 holds true for $\sigma = (M_k + C^{1/2})\delta_\varepsilon^{1/2}$ and $\varepsilon \leqslant \varepsilon_k$. We choose $\varepsilon_k^* > 0$, $\varepsilon_k \geqslant \varepsilon_k^* > 0$, ε_k^* sufficiently small in order to satisfy the condition $(\varkappa_k + 1)(\varrho + (2+\sigma)\sigma) < 1$ at $\sigma = (M_k + C^{1/2})\delta_\varepsilon^{1/2}$ and $\varrho = \widetilde{C}\delta_\varepsilon^{1/2}$. Then, by Lemma 2.2, (4.15) holds for $\widetilde{M}_k = \varkappa_k(\widetilde{C} + (3 + (M_k + C^{1/2}))(M_k + C^{1/2}))$. □

The following theorems improve estimates (4.3) and (4.4). Their proofs rely on the application of Lemma 2.1 using the test functions obtained from the higher order terms of the asymptotic expansions in Section 3 (cf. (3.1)–(3.3)). Formulas for matrix $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$, $\mathbf{M}$ and $\mathbf{N}$ are those derived in Section 3, namely (3.18), (3.27), (3.28), (3.12) and (3.17) respectively. The same can be said for their eigenvalues and eigenvectors.

Case where $\mu^0 \in \sigma(P_\Omega)$, with multiplicity $\kappa_\Omega(\mu^0) = q \geqslant 1$, and $\mu^0 \notin \sigma(P_\Gamma)$.

Theorem 4.3. *Let $\mu^0 = \mu_k^0$ be an eigenvalue of problem* (2.15) *which is not an eigenvalue of problem* (2.16); *$\mu_k^0 = \cdots = \mu_{k+q-1}^0$. Let μ^1 be an eigenvalue of the matrix $\mathbf{M} = (\int_\Gamma \partial_\nu V^i w_{1/2}^k \, \mathrm{d}\tau)_{i,k=1,\ldots,q}$ where $V^1, \ldots, V^q$ are the eigenfunctions of* (2.15) *associated with μ^0, orthonormal in $L^2(\Omega)$, and $w_{1/2}^i$ are the functions defined by* (3.9) *for* $i=1,\dots,q$. *Let* $\alpha=(\alpha_1,\dots,\alpha_q)^{\mathrm{T}}$ *be an eigenvector associated with* μ^1 *such that* $\sum_{i=1}^q|\alpha_i|^2=1$. *Let us consider* $V_0=\alpha_1V^1+\cdots+\alpha_qV^q$, $w_{1/2}=\alpha_1w^1_{1/2}+\cdots+\alpha_qw^q_{1/2}$ *and* V_1 *a solution of* (3.10) *where* $\mu^{1/2}=0$; *let* $w_{3/2}$ *be the solution of*:

$$\partial_\tau(h\partial_\tau w_{3/2})+\mu^0hw_{3/2}+\mu^1hw_{1/2}=\partial_\nu V_1. \tag{4.16}$$

Then, there exist eigenvalues μ^ε *of* (2.7), $\mu^\varepsilon=\mu^\varepsilon_j$ *for some* $j=k,\dots,k+q-1$, *such that for* ε *sufficiently small, namely* $\varepsilon<\varepsilon_*$,

$$\left|\mu^\varepsilon-\mu^0-\delta_\varepsilon\mu^1\right|<C_*\delta_\varepsilon^2, \tag{4.17}$$

where ε_* *and* C_* *are constants depending on* $\mu^0,\mu^1,V_1,w_{3/2}$ *and independent of* ε. *In addition, there are* $(\widetilde{\mathbf{U}}^\varepsilon,\tilde{\mathbf{w}}^\varepsilon)\in H^1(\Omega)\times H^1(\Gamma)$, $|||(\widetilde{\mathbf{U}}^\varepsilon,\tilde{\mathbf{w}}^\varepsilon)|||=1$, *each* $(\widetilde{\mathbf{U}}^\varepsilon,\tilde{\mathbf{w}}^\varepsilon)$ *belonging to the eigenspace associated with* $\mu^\varepsilon_{p(\varepsilon)}$ *of* (2.7) *which satisfy* $\mu^\varepsilon_{p(\varepsilon)}\in[\mu^0+\delta_\varepsilon\mu^1-K\delta_\varepsilon^\theta,\mu^0+\delta_\varepsilon\mu^1+K\delta_\varepsilon^\theta]$ *with* $K>0$ *and* $0<\theta<2$, *and* $(\widetilde{\mathbf{U}}^\varepsilon,\tilde{\mathbf{w}}^\varepsilon)$ *such that,*

$$\left|\left|\left|\left(\widetilde{\mathbf{U}}^\varepsilon,\tilde{\mathbf{w}}^\varepsilon\right)-\beta^\varepsilon\left(V_0+\delta_\varepsilon V_1+\delta_\varepsilon^2U_{w_{3/2}},\delta_\varepsilon^{1/2}w_{1/2}+\delta_\varepsilon^{3/2}w_{3/2}\right)\right|\right|\right|\leqslant\widetilde{C}_*\delta_\varepsilon^{2-\theta}, \tag{4.18}$$

where $\beta^\varepsilon=|||(V_0+\delta_\varepsilon V_1+\delta_\varepsilon^2U_{w_{3/2}},\delta_\varepsilon^{1/2}w_{1/2}+\delta_\varepsilon^{3/2}w_{3/2})|||^{-1}$ *and* $U_{w_{3/2}}\in H^1(\Omega)$ *is the function defined by* (4.1).

Proof. In a first stage we prove that there exists at least one eigenvalue $\mu^\varepsilon=\mu^\varepsilon_{p(\varepsilon)}$ of (2.7) satisfying (4.17) and there are $(\widetilde{\mathbf{U}}^\varepsilon,\tilde{\mathbf{w}}^\varepsilon)$ as the theorem states.

We consider the Hilbert space $\mathcal{H}_\varepsilon$ defined by (2.5) and the positive, selfadjoint and compact operator A_ε defined by (4.5), its eigenvalues being $\{(1+\mu^\varepsilon_k)^{-1}\}_{k=0}^\infty$. Let $\mu^0,\mu^1,V_0,V_1,w_{1/2},w_{3/2},U_{w_{3/2}}$ be as the theorem states. For sufficiently small ε, we consider the function $(V^\varepsilon,v^\varepsilon)=(V_0+\delta_\varepsilon V_1+\delta_\varepsilon^2U_{w_{3/2}},\delta_\varepsilon^{1/2}w_{1/2}+\delta_\varepsilon^{3/2}w_{3/2})$. It is clear that $(V^\varepsilon,v^\varepsilon)\in\mathcal{H}_\varepsilon$. In order to apply Lemma 2.1, we prove estimate:

$$\left|\left(\left(A_\varepsilon\left(\widetilde{V}^\varepsilon,\tilde{v}^\varepsilon\right)-\frac{1}{1+\mu^0+\delta_\varepsilon\mu^1}\left(\widetilde{V}^\varepsilon,\tilde{v}^\varepsilon\right),(G,g)\right)\right)\right|\leqslant C_*\delta_\varepsilon^2|||(G,g)|||\quad\forall(G,g)\in\mathcal{H}_\varepsilon, \tag{4.19}$$

where $(\widetilde{V}^\varepsilon,\tilde{v}^\varepsilon)=|||(V^\varepsilon,v^\varepsilon)|||^{-1}(V^\varepsilon,v^\varepsilon)$ and C_* is a constant independent of ε.

The definition of A_ε, $(V^\varepsilon,v^\varepsilon)$ and the scalar product $((\cdot,\cdot))$ yield

$$\left(1+\mu^0+\delta_\varepsilon\mu^1\right)\left(\left(A_\varepsilon\left(V^\varepsilon,v^\varepsilon\right)-\frac{1}{1+\mu^0+\delta_\varepsilon\mu^1}\left(V^\varepsilon,v^\varepsilon\right),(G,g)\right)\right)=J_1+J_2+J_3+J_4+J_5,$$

where

$$J_1=\mu^0\int_\Omega V_0G\,\mathrm{d}x+\int_\Omega\Delta V_0G\,\mathrm{d}x,$$

$$J_2=\delta_\varepsilon^{1/2}\mu^0\int_\Gamma w_{1/2}g\,\mathrm{d}\tau+\delta_\varepsilon^{1/2}\int_\Gamma\partial_\tau(h\partial_\tau w_{1/2})g\,\mathrm{d}\tau-\int_\Gamma\partial_\nu V_0G\,\mathrm{d}\tau,$$

$$J_3=\delta_\varepsilon\mu^1\int_\Omega V_0G\,\mathrm{d}x+\delta_\varepsilon\mu^0\int_\Omega V_1G\,\mathrm{d}x+\delta_\varepsilon\int_\Omega\Delta V_1G\,\mathrm{d}x,$$

$$J_4=\delta_\varepsilon^{3/2}\mu^0\int_\Gamma w_{3/2}g\,\mathrm{d}\tau+\delta_\varepsilon^{3/2}\mu^1\int_\Gamma w_{1/2}g\,\mathrm{d}\tau+\delta_\varepsilon^{3/2}\int_\Gamma\partial_\tau(h\partial_\tau w_{3/2})g\,\mathrm{d}\tau-\delta_\varepsilon\int_\Gamma\partial_\nu V_1G\,\mathrm{d}\tau\quad\text{and}$$

$$J_5=\delta_\varepsilon^2\mu^1\int_\Omega V_1G\,\mathrm{d}x+\delta_\varepsilon^2\left(\mu^0+\delta_\varepsilon\mu^1\right)\int_\Omega U_{w_{3/2}}G\,\mathrm{d}x+\delta_\varepsilon^{5/2}\mu^1\int_\Gamma hw_{3/2}g\,\mathrm{d}\tau-\delta_\varepsilon^2\int_\Omega\nabla_xU_{w_{3/2}}\cdot\nabla_xG\,\mathrm{d}x.$$

Now, the fact that $(G,g)\in\mathcal{H}_\varepsilon$ and that $V_0,V_1,w_{1/2},w_{3/2}$ are solutions of (2.15), (3.10), (3.7) and (4.16) respectively lead us to $J_1=J_2=J_3=J_4=0$. Moreover, $|J_5|\leqslant C_*\delta_\varepsilon^2|||(G,g)|||$ where C_* is a constant depending on $\mu^0,\mu^1,V_1,w_{3/2}$ but independent of ε. Finally, we have $|||(V^\varepsilon,v^\varepsilon)|||\to|||(V_0,0)|||=1+\mu^0$ as $\varepsilon\to0$, and estimate (4.19) holds for ε sufficiently small, $\varepsilon<\varepsilon_*$.

We apply Lemma 2.1 for $H = \mathcal{H}_\varepsilon$, $A = A_\varepsilon$, $\lambda = (1+\mu^0+\delta_\varepsilon\mu^1)^{-1}$, $u = (\widetilde{V}^\varepsilon, \tilde{v}^\varepsilon)$ and $r = \delta_\varepsilon^2 C_*$ which provides, for $\varepsilon < \varepsilon_*$, at least one eigenvalue $\mu^\varepsilon_{p(\varepsilon)}$ of (2.7) verifying $|(1+\mu^\varepsilon_{p(\varepsilon)})^{-1} - (1+\mu^0+\delta_\varepsilon\mu^1)^{-1}| \leqslant C_*\delta_\varepsilon^2$. We deduce (4.17). Moreover, if we take, for instance $r^* = \delta_\varepsilon^\theta$ with $0 < \theta < 2$, for $\varepsilon < \varepsilon_*$, Lemma 2.1 also provides a function $(\widetilde{\mathbf{U}}^\varepsilon, \tilde{\mathbf{w}}^\varepsilon) \in \mathcal{H}_\varepsilon$, with $|||(\widetilde{\mathbf{U}}^\varepsilon, \tilde{\mathbf{w}}^\varepsilon)||| = 1$, $(\widetilde{\mathbf{U}}^\varepsilon, \tilde{\mathbf{w}}^\varepsilon)$ belonging to the eigenspace associated with all the eigenvalues $(1+\mu^\varepsilon_{p(\varepsilon)})^{-1}$ of operator A_ε contained in $[(1+\mu^0+\delta_\varepsilon\mu^1)^{-1} - \delta_\varepsilon^\theta, (1+\mu^0+\delta_\varepsilon\mu^1)^{-1} + \delta_\varepsilon^\theta]$, such that (4.18) is satisfied.

In this way, the assertion performed at the beginning of the proof holds. Then, to conclude the proof of the theorem, we check that the eigenvalue $\mu^\varepsilon = \mu^\varepsilon_{p(\varepsilon)}$ verifying (4.17) is one of the eigenvalues μ^ε_j with $j = k, \ldots, k+q-1$. We show this assertion by contradiction.

Denoting by $\{\mu^\varepsilon_j\}_{j=k}^{j=k+q-1}$ the q eigenvalues provided by Theorem 4.1 which converge towards μ^0_k, we show that, for $\varepsilon < \varepsilon_*$, the eigenvalue $\mu^\varepsilon = \mu^\varepsilon_{p(\varepsilon)}$ obtained above coincides with one of the eigenvalues μ^ε_j with $j = k, \ldots, k+q-1$. First, we consider the case where the function $p(\varepsilon)$ is not bounded. Thus, there exists a subsequence ε' such that $p(\varepsilon') \to \infty$ as $\varepsilon' \to 0$. Consequently, for ε' sufficiently small, $p(\varepsilon') > k+q$ and $\mu^{\varepsilon'}_{p(\varepsilon')} \geqslant \mu^{\varepsilon'}_{k+q}$. Since $\mu^{\varepsilon'}_{p(\varepsilon')} \to \mu^0 = \mu^0_k$ and $\mu^{\varepsilon'}_{k+q} \to \mu^0_{k+q}$ as $\varepsilon' \to 0$, $\mu^0_k \geqslant \mu^0_{k+q}$ which contradicts the hypotheses in the theorem. Therefore, for ε sufficiently small, $p(\varepsilon)$ is bounded by some constant independent of ε.

Secondly, we consider that there exists a fixed l, $l \neq k, k+1, \ldots, k+q-1$, and a subsequence ε' such that $\mu^{\varepsilon'}_{p(\varepsilon')} = \mu^{\varepsilon'}_l$. Then, $\mu^{\varepsilon'}_{p(\varepsilon')} = \mu^{\varepsilon'}_l \to \mu^0_l$ as $\varepsilon' \to 0$, but $\mu^{\varepsilon'}_{p(\varepsilon')} \to \mu^0 = \mu^0_k$ which again contradicts the hypotheses in the theorem. Therefore, for $\varepsilon < \varepsilon_*$, the eigenvalue $\mu^\varepsilon = \mu^\varepsilon_{p(\varepsilon)}$ coincides with one of the eigenvalues μ^ε_j with $j = k, \ldots, k+q-1$ (obviously, the j can depend on ε) and the theorem is proved. □

Case where $\mu^0 \in \sigma(P_\Gamma)$, with multiplicity $\kappa_\Gamma(\mu^0) = p \geqslant 1$, and $\mu^0 \notin \sigma(P_\Omega)$.

Theorem 4.4. *Let $\mu^0 = \mu^0_k$ be an eigenvalue of problem* (2.16) *which is not an eigenvalue of problem* (2.15)*; $\mu^0_k = \cdots = \mu^0_{k+p-1}$. Let μ^1 be an eigenvalue of the matrix $\mathbf{N} = (\int_\Gamma \partial_\nu V^k_{1/2} w^j \,\mathrm{d}\tau)_{j,k=1,\ldots,p}$ where $w^1, \ldots, w^p$ are the eigenfunctions of* (2.16) *associated with μ^0, orthonormal in $L^2_h(\Gamma)$, and $V^j_{1/2}$ are the functions defined by* (3.16) *for $j = 1, \ldots, p$. Let $d = (d_1, \ldots, d_p)^{\mathrm{T}}$ be an eigenvector associated with μ^1 such that $\sum_{j=1}^p |d_j|^2 = 1$. Let us consider $w_0 = d_1 w^1 + \cdots + d_p w^p$, $V_{1/2} = d_1 V^1_{1/2} + \cdots + d_p V^p_{1/2}$ and w_1 a solution of* (3.11) *where $\mu^{1/2} = 0$; let $V_{3/2}$ be the solution of*:

$$\begin{cases} -\Delta_x V_{3/2} = \mu^0 V_{3/2} + \mu^1 V_{1/2} & \text{in } \Omega, \\ V_{3/2} = w_1 & \text{on } \Gamma. \end{cases}$$

Then, there exist eigenvalues μ^ε of (2.7)*, $\mu^\varepsilon = \mu^\varepsilon_j$ for some $j = k, \ldots, k+p-1$, such that for ε sufficiently small, namely $\varepsilon < \varepsilon_*$,*

$$\left|\mu^\varepsilon - \mu^0 - \delta_\varepsilon \mu^1\right| < C_* \delta_\varepsilon^2.$$

Moreover, there are $(\widetilde{\mathbf{U}}^\varepsilon, \tilde{\mathbf{w}}^\varepsilon) \in H^1(\Omega) \times H^1(\Gamma)$, $|||(\widetilde{\mathbf{U}}^\varepsilon, \tilde{\mathbf{w}}^\varepsilon)||| = 1$, each $(\widetilde{\mathbf{U}}^\varepsilon, \tilde{\mathbf{w}}^\varepsilon)$ belonging to the eigenspace associated with $\mu^\varepsilon_{p(\varepsilon)}$ of (2.7) *which satisfy $\mu^\varepsilon_{p(\varepsilon)} \in [\mu^0 + \delta_\varepsilon\mu^1 - K\delta_\varepsilon^\theta, \mu^0 + \delta_\varepsilon\mu^1 + K\delta_\varepsilon^\theta]$ with $K > 0$ and $0 < \theta < 2$, and $(\widetilde{\mathbf{U}}^\varepsilon, \tilde{\mathbf{w}}^\varepsilon)$ such that*

$$\left|\left|\left|(\widetilde{\mathbf{U}}^\varepsilon, \tilde{\mathbf{w}}^\varepsilon) - \beta^\varepsilon\left(\delta_\varepsilon^{1/2} V_{1/2} + \delta_\varepsilon^{3/2} V_{3/2}, w_0 + \delta_\varepsilon w_1\right)\right|\right|\right| \leqslant \widetilde{C}_* \delta_\varepsilon^{2-\theta}$$

where $\beta^\varepsilon = |||(\delta_\varepsilon^{1/2} V_{1/2} + \delta_\varepsilon^{3/2} V_{3/2}, w_0 + \delta_\varepsilon w_1)|||^{-1}$.

Proof. We use the technique in Theorem 4.3 for the test functions $(V^\varepsilon, v^\varepsilon) = (\delta_\varepsilon^{1/2} V_{1/2} + \delta_\varepsilon^{3/2} V_{3/2}, w_0 + \delta_\varepsilon w_1) \in \mathcal{H}_\varepsilon$ where $\mu^0, \mu^1, V_{1/2}, V_{3/2}, w_0, w_1$ are those in the statement of the theorem. By rewriting the reasoning in proof of Theorem 4.3 with minor modifications, Theorem 4.4 is proved. □

Resonance case, $\mu^0 \in \sigma(P_\Omega) \cap \sigma(P_\Gamma)$ with respective multiplicities $\kappa_\Omega(\mu^0) = q \geqslant 1$ and $\kappa_\Gamma(\mu^0) = p \geqslant 1$.

Theorem 4.5. *Let $\mu^0 = \mu_k^0$ be an eigenvalue of* (2.17), *$\mu^0 \in \sigma(P_\Omega) \cap \sigma(P_\Gamma)$, with multiplicity $\varkappa_k = p + q$; $\mu_k^0 = \cdots = \mu_{k+\varkappa_k-1}^0$. Let us consider $V^1, \ldots, V^q$ to be the eigenfunctions of* (2.15) *associated with μ^0, which are orthonormal in $L^2(\Omega)$, and let $w^1, \ldots, w^p$ be the corresponding eigenfunctions of* (2.16) *associated with μ^0, which are orthonormal in $L_h^2(\Gamma)$. Let us assume that the rank of the matrix $\mathbf{A} = (\int_\Gamma \partial_\nu V^i w^j \, d\tau)_{i=1,\ldots,q\ j=1,\ldots,p}$, is greater than zero. Let μ be a positive eigenvalue of the matrix $\mathbf{A}^{\mathrm{T}}\mathbf{A}$ and let $d = (d_1, \ldots, d_p)^{\mathrm{T}}$ be an eigenvector associated with μ such that $\sum_{j=1}^p |d_j|^2 = 1$. Let us consider $\mu^{1/2} = \pm\sqrt{\mu}$, $\alpha = (\mu^{1/2})^{-1}\mathbf{A}d$, $V_0 = \alpha_1 V^1 + \cdots + \alpha_q V^q$ and $w_0 = d_1 w^1 + \cdots + d_q w^q$. Let $V_{1/2}$ and $w_{1/2}$ be solutions of* (3.6) *and* (3.7) *respectively. Then, there exist eigenvalues μ^ε of* (2.7), *$\mu^\varepsilon = \mu_j^\varepsilon$ for some $j = k, \ldots, k + \varkappa_k - 1$, such that, for ε sufficiently small, namely $\varepsilon < \varepsilon_*$,*

$$\left|\mu^\varepsilon - \mu^0 - \delta_\varepsilon^{1/2}\mu^{1/2}\right| < C_*\delta_\varepsilon.$$

Moreover, there are $(\widetilde{\mathbf{U}}^\varepsilon, \tilde{\mathbf{w}}^\varepsilon) \in H^1(\Omega) \times H^1(\Gamma)$, $\|\!|(\widetilde{\mathbf{U}}^\varepsilon, \tilde{\mathbf{w}}^\varepsilon)|\!\| = 1$, each $(\widetilde{\mathbf{U}}^\varepsilon, \tilde{\mathbf{w}}^\varepsilon)$ belonging to the eigenspace associated with $\mu_{p(\varepsilon)}^\varepsilon$ of (2.7) *which satisfy $\mu_{p(\varepsilon)}^\varepsilon \in [\mu^0 + \delta_\varepsilon^{1/2}\mu^{1/2} - K\delta_\varepsilon^\theta, \mu^0 + \delta_\varepsilon^{1/2}\mu^{1/2} + K\delta_\varepsilon^\theta]$ with $K > 0$ and $0 < \theta < 1$, and $(\widetilde{\mathbf{U}}^\varepsilon, \tilde{\mathbf{w}}^\varepsilon)$ such that*

$$\left\|\!\left|(\widetilde{\mathbf{U}}^\varepsilon, \tilde{\mathbf{w}}^\varepsilon) - \beta^\varepsilon\left(V_0 + \delta_\varepsilon^{1/2}V_{1/2} + \delta_\varepsilon U_{w_{1/2}}, w_0 + \delta_\varepsilon^{1/2}w_{1/2}\right)\right|\!\right\| \leqslant \widetilde{C}_*\delta_\varepsilon^{1-\theta},$$

where $\beta^\varepsilon = \|\!|(V_0 + \delta_\varepsilon^{1/2}V_{1/2} + \delta_\varepsilon U_{w_{1/2}}, w_0 + \delta_\varepsilon^{1/2}w_{1/2})|\!\|^{-1}$ and $U_{w_{1/2}} \in H^1(\Omega)$ is the function defined by (4.1).

Proof. Similar arguments to the proof of Theorem 4.3 allow us to prove this theorem. Now, in order to apply Lemma 2.1 the test functions $(V^\varepsilon, v^\varepsilon)$ used are $(V^\varepsilon, v^\varepsilon) = (V_0 + \delta_\varepsilon^{1/2}V_{1/2} + \delta_\varepsilon U_{w_{1/2}}, w_0 + \delta_\varepsilon^{1/2}w_{1/2}) \in \mathcal{H}_\varepsilon$, where $V_0, V_{1/2}, U_{w_{1/2}}, w_0, w_{1/2}$ are defined in the statement of the theorem. □

Theorem 4.6. *Let $\mu^0 = \mu_k^0$ be an eigenvalue of* (2.17), *$\mu^0 \in \sigma(P_\Omega) \cap \sigma(P_\Gamma)$, with multiplicity $\varkappa_k = p + q$; $\mu_k^0 = \cdots = \mu_{k+\varkappa_k-1}^0$. Let us consider $V^1, \ldots, V^q$ to be the eigenfunctions of* (2.15) *associated with μ^0, which are orthonormal in $L^2(\Omega)$, and let $w^1, \ldots, w^p$ be the corresponding eigenfunctions of* (2.16) *associated with μ^0, which are orthonormal in $L_h^2(\Gamma)$. Let us assume that $q - n > 0$ where n is the rank of the matrix $\mathbf{A} = (\int_\Gamma \partial_\nu V^i w^j \, d\tau)_{i=1,\ldots,q,\ j=1,\ldots,p}$. Let μ^1 be an eigenvalue of the matrix $\mathbf{B} = (\sum_{i=1}^q \alpha_i^s \int_\Gamma \partial_\nu V^i w_{1/2}^k \, d\tau)_{s,k=1,\ldots,q-n}$ and let a be the corresponding eigenvector, where the vectors α^s verify* (3.20) *and $w_{1/2}^s$ is defined by* (3.22) *for $s = 1, \ldots, p - n$. Let us consider $\mu^{1/2} = 0$, $V_0 = \alpha_1 V^1 + \cdots + \alpha_q V^q$ and $w_{1/2} = \mathsf{a}_1 w_{1/2}^1 + \cdots + \mathsf{a}_{q-n} w_{1/2}^{q-n} + f_1 w^1 + \cdots + f_p w^p$ where α_j is defined by* (3.24) *for $j = 1, \ldots, q$, and f_k verifies* (3.25) *for $k = 1, \ldots, p$. Let V_1 be a solution of* (3.10). *Then, there exist eigenvalues μ^ε of* (2.7), *$\mu^\varepsilon = \mu_j^\varepsilon$ for some $j = k, \ldots, k + \varkappa_k - 1$, such that for ε sufficiently small, namely $\varepsilon < \varepsilon_*$,*

$$\left|\mu^\varepsilon - \mu^0 - \delta_\varepsilon\mu^1\right| < C_*\delta_\varepsilon^{3/2}.$$

Moreover, there are $(\widetilde{\mathbf{U}}^\varepsilon, \tilde{\mathbf{w}}^\varepsilon) \in H^1(\Omega) \times H^1(\Gamma)$, $\|\!|(\widetilde{\mathbf{U}}^\varepsilon, \tilde{\mathbf{w}}^\varepsilon)|\!\| = 1$, each $(\widetilde{\mathbf{U}}^\varepsilon, \tilde{\mathbf{w}}^\varepsilon)$ belonging to the eigenspace associated with $\mu_{p(\varepsilon)}^\varepsilon$ of (2.7) *which satisfy $\mu_{p(\varepsilon)}^\varepsilon \in [\mu^0 + \delta_\varepsilon\mu^1 - K\delta_\varepsilon^\theta, \mu^0 + \delta_\varepsilon\mu^1 + K\delta_\varepsilon^\theta]$ with $K > 0$ and $0 < \theta < 3/2$, and $(\widetilde{\mathbf{U}}^\varepsilon, \tilde{\mathbf{w}}^\varepsilon)$ such that*

$$\left\|\!\left|(\widetilde{\mathbf{U}}^\varepsilon, \tilde{\mathbf{w}}^\varepsilon) - \beta^\varepsilon\left(V_0 + \delta_\varepsilon V_1, \delta_\varepsilon^{1/2}w_{1/2}\right)\right|\!\right\| \leqslant \widetilde{C}_*\delta_\varepsilon^{3/2-\theta}$$

where $\beta^\varepsilon = \|\!|(V_0 + \delta_\varepsilon V_1, \delta_\varepsilon^{1/2}w_{1/2})|\!\|^{-1}$.

Proof. Similar reasonings to those used for the proof of Theorem 4.3 lead us to prove this theorem where $(V^\varepsilon, v^\varepsilon) = (V_0 + \delta_\varepsilon V_1, \delta_\varepsilon^{1/2}w_{1/2}) \in \mathcal{H}_\varepsilon$; $V_0, V_1, w_{1/2}$ as the theorem states. □

Theorem 4.7. *Let $\mu^0 = \mu_k^0$ be an eigenvalue of* (2.17), *$\mu^0 \in \sigma(P_\Omega) \cap \sigma(P_\Gamma)$, with multiplicity $\varkappa_k = p + q$; $\mu_k^0 = \cdots = \mu_{k+\varkappa_k-1}^0$. Let us consider $V^1, \ldots, V^q$ to be the eigenfunctions of* (2.15) *associated with μ^0, which are orthonormal in $L^2(\Omega)$, and let $w^1, \ldots, w^p$ be the corresponding eigenfunctions of* (2.16) *associated with μ^0, which are orthonormal in $L_h^2(\Gamma)$. Let us assume that $p - n > 0$ where n is the rank of the matrix $\mathbf{A} = (\int_\Gamma \partial_\nu V^i w^j \, d\tau)_{i=1,\ldots,q,\ j=1,\ldots,p}$. Let μ^1 be an eigenvalue of the matrix $\mathbf{C} = (\sum_{j=1}^p d_j^t \int_\Gamma \partial_\nu V_{1/2}^k w^j \, d\tau)_{t,k=1,\ldots,p-n}$ and let d be the corresponding eigenvector, where the vectors d^t verify* (3.19). *Let us consider $\mu^{1/2} = 0$, $w_0 = d_1 w^1 + \cdots + d_p w^p$ and*

$V_{1/2} = \mathsf{d}_1 V_{1/2}^1 + \cdots + \mathsf{d}_{p-n} V_{1/2}^{p-n} + \beta_1 V^1 + \cdots + \beta_q V^q$ where d_j is defined by (3.23) for $j = 1, \ldots, p$, $V_{1/2}^t$ is defined by (3.21) for $t = 1, \ldots, p-n$ and β_k verifies (3.26) for $k = 1, \ldots, q$. Let w_1 be a solution of (3.11). Then, there exist eigenvalues μ^ε of (2.7), $\mu^\varepsilon = \mu_j^\varepsilon$ for some $j = k, \ldots, k + \varkappa_k - 1$, such that, for ε sufficiently small, namely $\varepsilon < \varepsilon_*$,

$$\left|\mu^\varepsilon - \mu^0 - \delta_\varepsilon \mu^1\right| < C_* \delta_\varepsilon^{3/2}.$$

Moreover, there are $(\widetilde{\mathbf{U}}^\varepsilon, \tilde{\mathbf{w}}^\varepsilon) \in H^1(\Omega) \times H^1(\Gamma)$, $|||(\widetilde{\mathbf{U}}^\varepsilon, \tilde{\mathbf{w}}^\varepsilon)||| = 1$, each $(\widetilde{\mathbf{U}}^\varepsilon, \tilde{\mathbf{w}}^\varepsilon)$ belonging to the eigenspace associated with $\mu_{p(\varepsilon)}^\varepsilon$ of (2.7) which satisfy $\mu_{p(\varepsilon)}^\varepsilon \in [\mu^0 + \delta_\varepsilon \mu^1 - K\delta_\varepsilon^\theta, \mu^0 + \delta_\varepsilon \mu^1 + K\delta_\varepsilon^\theta]$ with $K > 0$ and $0 < \theta < 3/2$, and $(\widetilde{\mathbf{U}}^\varepsilon, \tilde{\mathbf{w}}^\varepsilon)$ such that

$$\left|\left|\left|\left(\widetilde{\mathbf{U}}^\varepsilon, \tilde{\mathbf{w}}^\varepsilon\right) - \beta^\varepsilon\left(\delta_\varepsilon^{1/2} V_{1/2} + \delta_\varepsilon^{3/2} U_{w_1}, w_0 + \delta_\varepsilon w_1\right)\right|\right|\right| \leqslant \widetilde{C}_* \delta_\varepsilon^{3/2-\theta},$$

where $\beta^\varepsilon = |||(\delta_\varepsilon^{1/2} V_{1/2} + \delta_\varepsilon^{3/2} U_{w_1}, w_0 + \delta_\varepsilon w_1)|||^{-1}$ and $U_{w_1} \in H^1(\Omega)$ is the function defined by (4.1).

Proof. By rewriting the reasoning in proof of Theorem 4.3 with $(V^\varepsilon, v^\varepsilon) = (\delta_\varepsilon^{1/2} V_{1/2} + \delta_\varepsilon^{3/2} U_{w_1}, w_0 + \delta_\varepsilon w_1) \in \mathcal{H}_\varepsilon$, the result in this theorem holds. □

Remark 4.1. Let us note that, in Theorems 4.1–4.7, the constants appearing in the estimates for the difference between the eigenelements of (2.13) involved with the kth eigenvalue μ_k^0, and those of the limiting problem (2.15)–(2.16), as well as the estimates involving the correcting terms, are obtained for ε smaller than a certain ε_k and they depend on k. Using the procedure of direct and inverse reduction (cf. [6,9,2]), we can specify these constants in terms of ε, k, and properties of the limiting spectrum, in an explicit way. As a sample, for the discrepancies involving leading terms (cf. (3.1)–(3.3)), we outline the results obtained using this technique in Theorem A.1 where, for the sake of brevity, we avoid proofs.

Remark 4.2. It should be noted that Theorems 4.3–4.7 justify results in Section 3 obtaining correcting terms which improve convergence in Theorems 4.1–4.2. The weaker bound for convergence rates obtained in Theorem 4.5 is in good agreement with Remark 3.1.

Also, we observe that the more precise results in these theorems correspond to the approach of the eigenvalues. As a matter of fact, for a fixed $j = k, \ldots, k + q - 1$ estimate (4.17) (and the analogues in Theorems 4.4–4.7) could hold for certain subsequences of ε. The statements of the results for eigenvalues and eigenfunctions definitely improve in the case where the limiting eigenvalue μ^0 is simple (cf. [7,11] for very different spectral problems).

5. Asymptotics for the stiff problem (1.2) in $\boldsymbol{\Omega_\varepsilon}$

In this section, we outline the asymptotic expansions postulated in [2] for the eigenpairs of (1.2) when $t > 1$ and prove certain results which are of interest for many problems of spectral perturbation theory in which the limiting problem of a self-adjoint parameter dependent problem proves to be non-selfadjoint, namely, Proposition 5.1. We also show how to modify these asymptotics in order to obtain the whole series for particular values of t. Finally, to avoid the non-selfadjoint limiting problem we provide alternative asymptotic expansions which lead us to a self-adjoint limiting problem. These results are justified in Section 6. Recall that for simplicity we set $A = a = 1$ in (1.2).

Taking into account estimate (2.9), we consider an asymptotic expansion for the eigenvalues λ^ε,

$$\lambda^\varepsilon = \mu + \mathrm{o}(1). \tag{5.1}$$

For the associated eigenfunctions, $\{U^\varepsilon, u^\varepsilon\}$, we can consider an asymptotic expansion in Ω and ω_ε, respectively,

$$U^\varepsilon(x) = V(x) + \mathrm{o}(1), \quad x \in \Omega, \tag{5.2}$$

$$u^\varepsilon(\zeta, \tau) = v_0(\zeta, \tau) + \mathrm{o}(1), \quad \zeta \in \left[0, h(\tau)\right), \tau \in \mathbb{S}_\ell, \tag{5.3}$$

where v_0 is a ℓ-periodic function in τ. Besides, we assume that the first term μ in (5.1) can be 0 while V in (5.2) or v_0 in (5.3) are different from zero. $\zeta = \varepsilon^{-1}\nu$ in (5.3), is the rapid variable. The terms o(1) in these asymptotic expansions mean further asymptotic terms, containing different order functions (powers of ε) which depend on t, and their dependence must be determined along with their respective accompanying terms. For simplicity, we consider only certain values of t, but we emphasize that the limiting problem satisfied by the leading terms in (5.1)–(5.3) is the same for any $t > 1$, namely, problem (5.7) (see Remark 5.1).

For instance, for natural t, $t > 1$, we consider an asymptotic expansion for the eigenvalues λ^ε and the associate eigenfunctions $\{U^\varepsilon, u^\varepsilon\}$:

$$\lambda^\varepsilon = \mu + \varepsilon\eta_1 + \varepsilon^2\eta_2 + \cdots, \tag{5.4}$$

$$U^\varepsilon(x) = V(x) + \varepsilon \mathsf{V}_1(x) + \varepsilon^2 \mathsf{V}_2(x) + \cdots, \quad x \in \Omega, \tag{5.5}$$

$$u^\varepsilon(\zeta,\tau) = v_0(\zeta,\tau) + \varepsilon v_1(\zeta,\tau) + \varepsilon^2 v_2(\zeta,\tau) + \cdots, \quad \zeta \in \big[0, h(\tau)\big),\ \tau \in \mathbb{S}_\ell, \tag{5.6}$$

where v_i are ℓ-periodic functions in τ. As usual, functions V, V_j and v_i, $j = 1, 2, \ldots$, $i = 0, 1, \ldots$ and the numbers μ and η_j are to be found by substitution in (1.2). We first determine the leading terms in (5.4)–(5.6).

By replacing expansions (5.4)–(5.6) in problem (1.2), after considering Eqs. (2.19) and (2.20), we collect coefficients of the same powers of ε and gather equations satisfied by V, v_j and μ (see [2] for more details). In a first step, we obtain the leading terms in (5.4)–(5.5) to satisfy the following equations:

$$\begin{cases} -\Delta_x V = \mu V & \text{in } \Omega, \\ \partial_\tau(h\partial_\tau V) + \mu h V = 0 & \text{on } \Gamma, \end{cases} \tag{5.7}$$

while the leading term in (5.6) is given by $v_0(\zeta,\tau) = V(0,\tau)$ for $\zeta \in [0, h(\tau))$, $\tau \in \mathbb{S}_\ell$.

Remark 5.1. Also replacing (5.1)–(5.3) in (1.2) leads us to the limiting problem (5.7). This is due to the fact that v_0, and the boundary condition in (5.7), are determined from the solvability condition for the Neumann problem satisfied by the term in (5.3) accompanying ε^2.

Let us note that the spectrum of problem (5.7) as a set is the union of the eigenvalues of two different problems, one of them posed in Ω, namely problem (2.15), and another on Γ, namely problem (2.16). Indeed, first we observe that, if $\mu \notin \sigma(P_\Gamma)$, then an eigenelement (μ, V) of problem (5.7) satisfies (2.15) and, of course, all eigenelements of (2.15) satisfy (5.7). On the other hand, in the case where $\mu \in \sigma(P_\Gamma)$ and $\mu \notin \sigma(P_\Omega)$, we extend the eigenfunction W to Ω by the unique solution of the Dirichlet problem:

$$\begin{cases} -\Delta_x V = \mu V & \text{in } \Omega, \\ V = W & \text{on } \Gamma, \end{cases} \tag{5.8}$$

in order to obtain an eigenfunction of problem (5.7) associated with μ.

Finally, in the case where μ is an eigenvalue of (2.15) with multiplicity $\kappa_\Omega(\mu) = q$, and of (2.16) with multiplicity $\kappa_\Gamma(\mu) = p$, $p \leqslant 2$, let us consider $V^1, \ldots, V^q$ the eigenfunctions of (2.15) associated with μ, which are orthonormal in $L^2(\Omega)$, and let $w^1, \ldots, w^p$ be p linearly independent eigenfunctions of (2.16) associated with μ, in $L^2(\Gamma)$. We use the Fredholm alternative to deduce that for each eigenfunction w^j of problem (2.16) satisfying the compatibility conditions,

$$\int_\Gamma \partial_\nu V^i w^j \, \mathrm{d}\tau = 0,$$

for $i = 1, \ldots, q$, can be extended on Ω as the solution V_{w^j} of the Dirichlet problem (5.8) which is unique in the orthogonal complement of the subspace $[V^1, \ldots, V^q]$ in $H^1(\Omega)$. In this way, we have proved that the multiplicity of any eigenvalue μ of (5.7), which is an eigenvalue of both problems (2.15) and (2.16) simultaneously, depends on the multiplicity of μ as an eigenvalue of (2.16), or as an eigenvalue of (2.15), and of the rank of the matrix $\mathbf{A} = (\int_\Gamma \partial_\nu V^i w^j \, \mathrm{d}\tau)_{i=1,\ldots,q,\, j=1,\ldots,p}$ as follows:

$$\kappa_g(\mu) = \kappa_\Omega(\mu) + \kappa_\Gamma(\mu) - \operatorname{rank}(\mathbf{A}).$$

Here $\kappa_g(\mu)$ denotes the geometric multiplicity of μ as an eigenvalue of problem (5.7). Next, we prove that, in general, problem (5.7) cannot be associated with a self-adjoint operator, since the unexpected fact established in the following proposition shows that the algebraic multiplicity $\kappa_a(\mu)$ of the eigenvalue μ of (5.7) can be strictly greater than $\kappa_g(\mu)$.

Note that the matrix $\mathbf{A}$ has already been defined in (3.18); let n denote its rank. Without any loss of generality, we can assume that $V^1, \ldots, V^q$ and $w^1, \ldots, w^p$ are ordered in such a way that the left-hand and top $n \times n$-block of the

$q \times p$ matrix $\mathbf{A}$ is non-degenerated and $\kappa_g(\mu)$ linearly independent eigenfunctions of problem (5.7) associated with μ are

$$\{V^1, \dots, V^q, V_{w^{n+1}}, \dots, V_{w^p}\}. \tag{5.9}$$

In any case, we assume that the index out of those possible indexes for V^j or w^i means that we do not take into account the corresponding functions.

Proposition 5.1. *Problem* (5.7) *has exactly n associated functions $\mathbb{V}_1^1, \dots, \mathbb{V}_1^n$ corresponding to n linearly independent eigenfunctions $\widetilde{V}^1, \dots, \widetilde{V}^n$ of* (2.15) *associated with the eigenvalue μ. In addition, the Jordan chains $\{\widetilde{V}^j, \mathbb{V}_1^j\}$, $j = 1, \dots, n$ here constructed, cannot be extended.*

Proof. The problem of finding associated functions reads

$$\begin{cases} -\Delta_x \mathbb{V}_1 = \mu \mathbb{V}_1 + V & \text{in } \Omega, \\ \partial_\tau (h \partial_\tau \mathbb{V}_1) + \mu h \mathbb{V}_1 + hV = 0 & \text{on } \Gamma, \end{cases} \tag{5.10}$$

where V is an eigenfunction of (5.7), namely, V belongs to the linear space generated by (5.9). First, we observe that if $V \in [V_{w^{n+1}}, \dots, V_{w^p}]$, then the equation on Γ in (5.10) has no solution. The same holds for the case where V belongs to the eigenspace $[V^1, \dots, V^q, V_{w^{n+1}}, \dots, V_{w^p}]$ with $V \neq 0$ on Γ; this is due to the fact that in this case the restriction of V to Γ is a linear combination of eigenfunctions w^j for $j = n+1, \dots, p$. Thus, for both cases, there are no associated functions and the only possibility of finding associated functions is that $V \in [V^1, \dots, V^q]$.

Let us consider the last case. That is, the case where

$$V = \beta_1 V^1 + \cdots + \beta_q V^q, \tag{5.11}$$

for certain constants $\beta_1, \dots, \beta_q$. Since $V = 0$ on Γ, the equation for $\mathbb{V}_1$ on Γ in (5.10) is homogeneous and therefore

$$\mathbb{V}_1 = \alpha_1 w^1 + \cdots + \alpha_p w^p \quad \text{on } \Gamma,$$

for certain constants $\alpha_1, \dots, \alpha_p$. From the orthogonality of the eigenfunctions $\{V^1, \dots, V^q\}$ in $L^2(\Omega)$, and the compatibility conditions for the Dirichlet problem in Ω we deduce:

$$\beta_j = \int_\Omega V V^j \,\mathrm{d}x = \sum_{s=1}^p \alpha_s \int_\Gamma \partial_\nu V^j w^s \,\mathrm{d}\tau, \quad j = 1, \dots, q. \tag{5.12}$$

Next, we show that these equations can be fulfilled by a proper choice of the coefficients α_s and β_j, depending on n.

For $n = 0$, (5.12) gives $\beta_1 = \cdots = \beta_q = 0$; therefore, in this case there are no associated functions corresponding to the eigenvalue μ and the proposition is proved.

For $n \geqslant 1$ and for $k = 1, \dots, n$, considering the structure of the matrix $\mathbf{A}$, in (5.12) we can choose $\alpha_s \equiv \alpha_s^k = \delta_{s,k}$, for $s = 1, \dots, p$, which gives n vectors $\beta = (\beta_1, \dots, \beta_q)$ as the n first columns of the matrix $\mathbf{A}$, and therefore n linearly independent vectors β of $\mathbb{R}^q$. Then, (5.11) provides at least n eigenfunctions of (5.7) which have associated functions, i.e., $\mathbb{V}_1^k$ solutions of (5.10) for $V = \widetilde{V}^k$ in (5.11). Obviously $V^1, \dots, V^q, V_{w^{n+1}}, \dots, V_{w^p}, \mathbb{V}_1^1, \dots, \mathbb{V}_1^n$ are $p + q$ linearly independent functions in $H^1(\Omega)$. Moreover, since the rank of $\mathbf{A}$ is n, the maximum number of linearly independent vectors $\beta = (\beta_1, \dots, \beta_q)$ of $\mathbb{R}^q$ given by (5.12) is n, and therefore, the above argument provides all the possible associated functions corresponding to the eigenvalue μ.

On the other hand, since $|\alpha_1| + \cdots + |\alpha_p| > 0$, the problem:

$$\begin{cases} -\Delta_x \mathbb{V}_2 = \mu \mathbb{V}_2 + \mathbb{V}_1 & \text{in } \Omega, \\ \partial_\tau (h \partial_\tau \mathbb{V}_2) + \mu h \mathbb{V}_2 + h\mathbb{V}_1 = 0 & \text{on } \Gamma, \end{cases}$$

for associated functions of second order has no solution, and the proof of the proposition is completed for $V \equiv \widetilde{V}^j$ defined by (5.11) and $\mathbb{V}_1^j$ the associated functions satisfying (5.10). □

We observe that depending on the problem, Proposition 5.1 ensures that there can exist associated functions. As a direct consequence, and by analogy with the case of non-self adjoint operators on Hilbert spaces, we can claim that the total algebraic multiplicity of the eigenvalue μ of problem (5.7) is,

$$\kappa_a(\mu) = \kappa_\Omega(\mu) + \kappa_\Gamma(\mu).$$

We also note that the appearance of associated functions in the limiting problem (5.7) seems to be rather surprising since the original ε-dependent problem is self-adjoint.

As a sample, for $t = 2$ we show below that the process with the same ansätze (5.4)–(5.6) cannot always be extended in the resonant case where μ is an eigenvalue of both problems (2.15) and (2.16). In this last case, in general, expansions (5.4)–(5.6) are not consistent since (5.7) is not formally self-adjoint, and, in order to extend the process, we need to modify suitably these expansions, depending on the multiplicity of the eigenvalue μ, introducing different powers of ε (cf. [8] and Section V.3 in [12] for related questions). In order to show the process, we consider the simplest case, namely, *the resonant case where μ is a simple eigenvalue of both problems* (2.15) *and* (2.16), the corresponding eigenfunctions V and W satisfying:

$$\int_{\Gamma} \partial_\nu V W \,\mathrm{d}\tau = I \neq 0. \tag{5.13}$$

Further terms of asymptotic expansions (5.4)–(5.6) for $t = 2$

Now, assuming that μ is a simple eigenvalue of (5.7), the process used to find the leading terms (μ, V) and v_0 in the asymptotic expansions should be continued to determine the other terms of (5.4)–(5.6) always depending on the fixed t. We obtain that the second terms η_1 and V_1 in (5.4) and (5.5) verify the non homogeneous problem, associated with (5.7),

$$\begin{cases} -\Delta_x \mathsf{V}_1 - \mu \mathsf{V}_1 = \eta_1 V & \text{in } \Omega, \\ \partial_\tau (h \partial_\tau \mathsf{V}_1) + \mu h \mathsf{V}_1 = \partial_\nu V + F_V - \eta_1 h V & \text{on } \Gamma, \end{cases} \tag{5.14}$$

where F_V is the function defined on Γ by:

$$F_V(\tau) = \frac{1}{2} \partial_\tau V(0,\tau) h(\tau)\big(3h'(\tau)\varkappa(\tau) + h(\tau)\varkappa'(\tau)\big) + \partial_\tau^2 V(0,\tau)\varkappa(\tau)h(\tau)^2, \quad \text{for } \tau \in \mathbb{S}_\ell,$$

while the second term in (5.6) is given by $v_1(\zeta,\tau) = \mathsf{V}_1(0,\tau)$ for $\zeta \in [0, h(\tau))$, $\tau \in \mathbb{S}_\ell$. Considering all the possible solutions (μ, V) to (5.7), the compatibility condition for (5.14) determines η_1 except in the case where μ is an eigenvalue of (2.15) and (2.16) simultaneously.

Since μ is a simple eigenvalue of (5.7), under the assumption (5.13), we have $q = p = n = 1$ in Proposition 5.1 and $\mathbb{V}_1$ the associated function with V. In this case, we show that it is possible to find eigenelements of (1.2) with asymptotics (cf. (3.1)–(3.3) with $\delta_\varepsilon = \varepsilon$ to compare),

$$\begin{gathered} \lambda^\varepsilon = \mu \pm \varepsilon^{1/2}\eta_{1/2} + \varepsilon \eta_1^\pm + \cdots, \\ U^\varepsilon(x) = V^0(x) + \varepsilon^{1/2}\mathsf{V}_{1/2}^\pm(x) + \varepsilon \mathsf{V}_1^\pm(x) + \cdots, \\ u^\varepsilon(\zeta,\tau) = w(\tau) + \varepsilon^{1/2}\mathsf{W}_{1/2}^\pm(\zeta,\tau) + \varepsilon \mathsf{W}_1^\pm(\zeta,\tau) + \cdots. \end{gathered}$$

Indeed, inserting these ansätze into (1.2), we readily find that the pair $(V^0, w(\tau))$ satisfies (5.7) and in this case, where μ is a simple eigenvalue of both problems (2.15) and (2.16), $V^0 = V$, $w = 0$ and the problem for the first correction terms reads:

$$\begin{cases} -\Delta_x \mathsf{V}_{1/2} - \mu \mathsf{V}_{1/2} = \pm \eta_{1/2} V & \text{in } \Omega, \\ \partial_\tau (h \partial_\tau \mathsf{V}_{1/2}) + \mu h \mathsf{V}_{1/2} = \mp \eta_{1/2} h V = 0 & \text{on } \Gamma, \end{cases}$$

which admits the solution

$$\mathsf{V}_{1/2}^\pm(x) = \pm \eta_{1/2}\mathbb{V}_1(x) + a_{1/2} V(x), \qquad \mathsf{W}_{1/2}^\pm(\zeta,\tau) = \pm \eta_{1/2} I^{-1} W(\tau),$$

for $a_{1/2}$ a certain constant.

We continue the process to obtain the second correction terms; we have:

$$\begin{cases} -\Delta_x \mathsf{V}_1 - \mu \mathsf{V}_1 = \pm \eta_{1/2} \mathsf{V}_{1/2}^\pm + \eta_1 V & \text{in } \Omega, \\ \partial_\tau (h \partial_\tau \mathsf{V}_1) + \mu h \mathsf{V}_1 = \partial_\nu V + F_V \mp \eta_{1/2} h \mathsf{V}_{1/2}^\pm - \eta_1 h V & \text{on } \Gamma, \end{cases} \tag{5.15}$$

for the same function F_V in (5.14). Now, since $V = 0$ and $\mathsf{V}_{1/2}^\pm = \mathsf{W}_{1/2}^\pm$ on Γ, Eq. (5.15) on Γ takes the form,

$$\partial_\tau (h \partial_\tau \mathsf{V}_1) + \mu h \mathsf{V}_1 = \partial_\nu V - \eta_{1/2}^2 h I^{-1} W, \tag{5.16}$$

and has a solution if and only if

$$\int_\Gamma \partial_\nu V W \,\mathrm{d}\tau = \eta_{1/2}^2 I^{-1} \int_\Gamma h W^2 \,\mathrm{d}\tau = \eta_{1/2}^2 I^{-1}.$$

Thus, $\eta_{1/2} = I$. Hence, the solution of Eq. (5.16) can be chosen as follows:

$$\mathsf{V}_1 = \mathsf{W}_1^\bullet + b_1 W,$$

where $\mathsf{W}_1^\bullet$ is a particular solution and the constant b_1 is to be determined such that the Dirichlet problem in Ω for V_1,

$$\begin{cases} -\Delta_x \mathsf{V}_1 - \mu \mathsf{V}_1 = \pm\eta_{1/2}\mathsf{V}_{1/2}^{\pm} + \eta_1 V & \text{in } \Omega, \\ \mathsf{V}_1 = \mathsf{W}_1^\bullet + b_1 W & \text{on } \Gamma, \end{cases}$$

has a solution.

New asymptotic expansions after a re-scaling of the eigenfunctions in ω_ε

Note that the rare behavior noticed for the limit problem (5.7) can be explained if we observe that the local asymptotic expansion (5.6) may not be consistent with the normalization for the eigenfunctions (2.10). In fact, (2.10), (2.9), and (5.1)–(5.3) imply the first term in (5.6) $v_0 \equiv 0$, and therefore the leading terms μ and V in (5.4) and (5.5) can only be an eigenvalue and the associated eigenfunction of the Dirichlet problem (2.15).

Thus, it proves necessary either to change the normalization of the eigenfunctions in order to keep the right asymptotic expansions for the eigenfunctions in (5.5)–(5.6) or to change these asymptotic expansions in a consistent way with the normalization (2.10). In this last respect, considering the convergence results obtained in [2] for the different case where $t = 1$ in problem (1.2) (cf. also problem $(1.2)_1$, $(1.2)_3$–$(1.2)_5$, (1.3) when $t = 1$ and $m = 0$), it proves useful to consider the re-scaled eigenfunctions in ω_ε, $w^\varepsilon = \varepsilon^{-\frac{t-1}{2}} u^\varepsilon$.

Now, on account of the identification of the elements $\{U^\varepsilon, u^\varepsilon\} \in H^1(\Omega_\varepsilon)$ with the pairs $(U^\varepsilon, w^\varepsilon) \in \mathcal{V}_\varepsilon$ (cf. (2.11)) we modify the asymptotic expansions (5.1)–(5.3) and, more specifically, we assume a different expansion in ω_ε for the eigenfunctions:

$$w^\varepsilon(\zeta, \tau) = w_0(\zeta, \tau) + \mathrm{o}(1), \quad \zeta \in \big[0, h(\tau)\big),\ \tau \in \mathbb{S}_\ell. \tag{5.17}$$

Then, performing the change $u^\varepsilon = \varepsilon^{\frac{t-1}{2}} w^\varepsilon$ in (1.2), and replacing expansions (5.1), (5.2) and (5.17) in this problem, on account of Eqs. (2.19) and (2.20), we collect coefficients of the same powers of ε and gather equations satisfied by V, w_0 and μ. In a first step, we have that the leading terms μ, V and w_0 in these expansions satisfy problems (2.15) or (2.16). Here w_0 is determined as in Remark 5.1.

6. Convergence for the eigenelements of problem (2.13)

In this section we justify the asymptotic expansions for the eigenelements of (1.2) in Section 5, and, more precisely, those derived from the ansätze (5.1), (5.2) and (5.17). That is, we prove the convergence of the eigenelements of (2.13) towards those of (2.17). In addition, in Section 6.1 we justify the connection between problems (2.13) and (2.7) (i.e., problem (1.2) for $t > 1$ and (1.1)) comparing the spectra. Specifying, Theorems 6.2 and 6.3 show that the eigenelements of (1.1) provide an alternative or even better approach to those of (1.2) than the eigenelements of the limiting problem (2.15)–(2.16) (see Theorems 6.1 and A.2 and Remark A.1 to compare).

The first result on spectral convergence in this section is obtained using Lemma 2.3 and it is stated in Theorem 6.1. Also the convergence for the eigenfunctions holds. As in Section 4, bounds for the discrepancies of the eigenvalues and eigenfunctions in terms of constants depending on the eigenvalue number can be derived (cf. (4.3), (4.4) and (4.15) to compare). We avoid obtaining these bounds here, for the sake of brevity, since as a matter of fact, more precise bounds are stated in Theorem A.2.

Recall that throughout all the section, the parameter t takes values greater than 1 and the normalization for the eigenfunctions of (2.13) is given by (2.14). We also recall the extension operator from $H^1(\Gamma)$ into $H^1(\Omega)$ introduced in Section 4 satisfying (4.1) and (4.2). In addition, for proofs in this section, we introduce a new extension operator R_ε from $H_0^1(\Omega) \times H^1(\Gamma)$ into $\mathcal{V}_\varepsilon$ as follows:

Let us consider the linear operator $\mathsf{R}_\varepsilon : H_0^1(\Omega) \times H^1(\Gamma) \to \mathcal{V}_\varepsilon$ defined by $\mathsf{R}_\varepsilon(F, f) = (\check{F}^\varepsilon, \check{f}^\varepsilon)$ for any $(F, f) \in H_0^1(\Omega) \times H^1(\Gamma)$, where

$$\begin{aligned} \check{F}^\varepsilon(x) &= F(x) + \varepsilon^{(t-1)/2} U_f(x) \quad &\text{if } x \in \Omega, \\ \check{f}^\varepsilon(x) &= f(\tau) \quad &\text{if } x \in \omega_\varepsilon, \end{aligned} \tag{6.1}$$

with U_f the function defined by (4.1).

Theorem 6.1. *For fixed $t > 1$, and for each fixed k, $k = 0, 1, 2 \ldots$, the eigenvalues λ_k^ε of* (2.13) *converge towards the eigenvalue μ_k^0 of* (2.17) *as $\varepsilon \to 0$. Moreover, for any eigenvalue μ_k^0 of* (2.17) *with multiplicity $\varkappa_k$ ($\mu_k^0 = \mu_{k+1}^0 = \cdots = \mu_{k+\varkappa_k-1}^0$), and for any eigenfunction (V, w) of* (2.17) *associated with μ_k^0 with $|||(V, w)||| = 1$, there is a linear combination $(\widetilde{U}^\varepsilon, \tilde{w}^\varepsilon)$ of eigenfunctions associated with the eigenvalues $\{\lambda_i^\varepsilon\}_{i=k}^{k+\varkappa_k-1}$ converging towards μ_k^0, such that*

$$\left\| \left(\widetilde{U}^\varepsilon, \tilde{w}^\varepsilon\right) - \mathsf{R}_\varepsilon(V, w) \right\|_\varepsilon \xrightarrow{\varepsilon \to 0} 0. \tag{6.2}$$

In addition, for each sequence $(U_k^\varepsilon, w_k^\varepsilon)$ of eigenfunctions of (2.13), *$\|(U_k^\varepsilon, w_k^\varepsilon)\|_\varepsilon = 1$, we can extract a subsequence (still denoted by ε) such that $(U_k^\varepsilon, w_k^\varepsilon|_\Gamma) \to (V_k^*, w_k^*)$ in $H^1(\Omega) \times H^1(\Gamma)$-weak, as $\varepsilon \to 0$, where (V_k^*, w_k^*) is an eigenfunction of* (2.17) *associated with μ_k^0 and the set $\{(V_k^*, w_k^*)\}_{k=0}^\infty$ forms an orthonormal basis of $H_0^1(\Omega) \times H^1(\Gamma)$ for the scalar product* (2.6).

Proof. Let us consider the Hilbert space $\mathcal{V}_\varepsilon$ defined by (2.11) with the scalar product (2.12). Let B_ε be the positive, selfadjoint and compact operator defined on $\mathcal{V}_\varepsilon$ as follows: for any $(F, f) \in \mathcal{V}_\varepsilon$, $B_\varepsilon(F, f) = (U^\varepsilon, u^\varepsilon)$ where $(U^\varepsilon, u^\varepsilon) \in \mathcal{V}_\varepsilon$ is the unique solution of:

$$\left(\left(U^\varepsilon, u^\varepsilon\right), (G, g)\right)_\varepsilon = \int_\Omega F G \, \mathrm{d}x + \frac{1}{\varepsilon} \int_{\omega_\varepsilon} f g \, \mathrm{d}x \quad \forall (G, g) \in \mathcal{V}_\varepsilon. \tag{6.3}$$

Obviously, the eigenvalues of B_ε are $\{(1 + \lambda_k^\varepsilon)^{-1}\}_{k=0}^\infty$ where $\{\lambda_k^\varepsilon\}_{k=0}^\infty$ are the eigenvalues of (2.13).

In the same way, we consider the Hilbert space $\mathcal{H}_0 = H_0^1(\Omega) \times H^1(\Gamma)$ with the scalar product (2.6) and define the operator A_0 on $\mathcal{H}_0$ defined by (4.6) whose eigenvalues are $\{(1 + \mu_k^0)^{-1}\}_{k=0}^\infty$; that is, $A_0(F, f) = (U, u)$, for $(F, f) \in \mathcal{H}_0$, where (U, u) is the unique solution of (4.6).

Let $\mathcal{W}$ be the space $\mathcal{W} = H_0^1(\Omega) \times H^1(\Gamma)$ and let R_ε be the linear, continuous operator $\mathsf{R}_\varepsilon : \mathcal{H}_0 \to \mathcal{V}_\varepsilon$ defined by (6.1). We check the properties (a)–(d) of Lemma 2.3.

For any $(F, f) \in H^1(\Omega) \times H^1(\Gamma)$, let us consider $(\check{F}^\varepsilon, \check{f}^\varepsilon)$ defined by (6.1). By virtue of (6.1), (4.2), (2.21) and (2.22), we obtain:

$$\left\| \check{F}^\varepsilon - F \right\|_{H^1(\Omega)} \leqslant C \varepsilon^{(t-1)/2} \|f\|_{H^1(\Gamma)}, \tag{6.4}$$

$$\left| \frac{1}{\varepsilon} \left\| \check{f}^\varepsilon \right\|_{L^2(\omega_\varepsilon)}^2 - \|f\|_{L_h^2(\Gamma)}^2 \right| \leqslant C \varepsilon \|f\|_{L_h^2(\Gamma)}^2, \tag{6.5}$$

$$\left| \frac{1}{\varepsilon} \left\| \nabla_x \check{f}^\varepsilon \right\|_{L^2(\omega_\varepsilon)}^2 - \|\partial_\tau f\|_{L_h^2(\Gamma)}^2 \right| \leqslant C \varepsilon \|\partial_\tau f\|_{L_h^2(\Gamma)}^2,$$

where C is a constant independent of ε and t. Thus, for $(F, f) \in H_0^1(\Omega) \times H^1(\Gamma)$, $\|\mathsf{R}_\varepsilon(F, f)\|_\varepsilon^2 = \|(\check{F}^\varepsilon, \check{f}^\varepsilon)\|_\varepsilon^2 \to |||(F, f)|||^2$, when $\varepsilon \to 0$, and property (a) holds.

In order to prove the boundedness of $\|B_\varepsilon\|_{\mathcal{L}(\mathcal{V}_\varepsilon)}$, for any $(F, f) \in \mathcal{V}_\varepsilon$ we consider $B_\varepsilon(F, f) = (U^\varepsilon, u^\varepsilon)$ the solution of (6.3) and take $(G, g) = (U^\varepsilon, u^\varepsilon)$. Applying the Cauchy–Buniakowsky–Schwarz inequality, we get $\|(U^\varepsilon, u^\varepsilon)\|_\varepsilon \leqslant C\|(F, f)\|_\varepsilon$ where C is a constant independent of ε and (F, f). Thus, property (b) is satisfied.

Let us prove property (c). For each $\varepsilon > 0$ and any fixed $(F, f) \in \mathcal{H}_0$, we consider $(U^\varepsilon, u^\varepsilon) = B_\varepsilon \mathsf{R}_\varepsilon(F, f)$; $(U^\varepsilon, u^\varepsilon) \in \mathcal{V}_\varepsilon$ satisfies (6.3) for $(F, f) = (\check{F}^\varepsilon, \check{f}^\varepsilon)$, that is,

$$\left(\left(U^\varepsilon, u^\varepsilon\right), (G, g)\right)_\varepsilon = \int_\Omega \check{F}^\varepsilon G \, \mathrm{d}x + \frac{1}{\varepsilon} \int_{\omega_\varepsilon} \check{f}^\varepsilon g \, \mathrm{d}x \quad \forall (G, g) \in \mathcal{V}_\varepsilon. \tag{6.6}$$

On account of (6.4) and (6.5), we consider (6.6) with $(G,g)=(U^\varepsilon,u^\varepsilon)$ and we obtain that, for ε sufficiently small,

$$\left\|\left(U^\varepsilon,u^\varepsilon\right)\right\|_\varepsilon\leqslant C, \tag{6.7}$$

where C is a constant independent of ε.

For each $\varepsilon>0$, we introduce in (6.7) the change of variables in ω_ε from Cartesian coordinates x_1,x_2 to local coordinates (ζ,τ) where $\zeta=\nu/\varepsilon$ and (ν,τ) are the curvilinear coordinates. Then, (6.7) reads:

$$\left\|U^\varepsilon\right\|^2_{H^1(\Omega)}+\int_0^\ell\int_0^{h(\tau)}\left|\partial_\tau \mathsf{u}^\varepsilon\right|^2K_\varepsilon^{-1}\,\mathrm{d}\zeta\,\mathrm{d}\tau+\frac{1}{\varepsilon^2}\int_0^\ell\int_0^{h(\tau)}\left|\partial_\zeta \mathsf{u}^\varepsilon\right|^2K_\varepsilon\,\mathrm{d}\zeta\,\mathrm{d}\tau+\int_0^\ell\int_0^{h(\tau)}\left|\mathsf{u}^\varepsilon\right|^2K_\varepsilon\,\mathrm{d}\zeta\,\mathrm{d}\tau\leqslant C,$$

where $\mathsf{u}^\varepsilon(\zeta,\tau)$ denote the eigenfunctions $u^\varepsilon(x)$ in the local coordinates (ζ,τ) and $K_\varepsilon(\zeta,\tau)=1+\varepsilon\zeta\kappa(\tau)$. Taking into account that bounds (2.21) also hold for $K_\varepsilon(\zeta,\tau)$, $\forall\zeta\in[0,h(\tau)]$, $\tau\in\mathbb{S}_\ell$ and ε sufficiently small, we can write:

$$\left\|U^\varepsilon\right\|^2_{H^1(\Omega)}+\left\|\mathsf{u}^\varepsilon\right\|^2_{L^2(\Pi)}+\left\|\partial_\tau\mathsf{u}^\varepsilon\right\|^2_{L^2(\Pi)}+\varepsilon^{-2}\left\|\partial_\zeta\mathsf{u}^\varepsilon\right\|^2_{L^2(\Pi)}\leqslant C, \tag{6.8}$$

with Π the domain $(0,h(\tau))\times(0,\ell)$. Therefore, we can extract a subsequence, still denoted by ε, such that $(U^\varepsilon,\mathsf{u}^\varepsilon)$ converges, as $\varepsilon\to0$, towards some function (U^*,u^*) weakly in $H^1(\Omega)\times H^1(\Pi)$. Since $(U^\varepsilon,u^\varepsilon)\in\mathcal{V}_\varepsilon$, $U^\varepsilon=\varepsilon^{(t-1)/2}\mathsf{u}^\varepsilon$ on Γ; hence $U^*=0$ on Γ which ensures $U^*\in H^1_0(\Omega)$. Moreover, from (6.8) it follows that $\|\partial_\zeta\mathsf{u}^\varepsilon\|^2_{L^2(\Pi)}\leqslant C\varepsilon^2$ and consequently $\partial_\zeta\mathsf{u}^*=0$ in Π; thus, it does not depend on ζ and we can write $\mathsf{u}^*=u^*(\tau)$ in Π for a certain function $u^*(\tau)$.

In order to identify (U^*,u^*), for any $(V,v)\in\mathcal{H}_0$ fixed, we consider (6.6) for $(G,g)=\mathsf{R}_\varepsilon(V,v)$ and introduce the local coordinates (ζ,τ) in the domain ω_ε:

$$\begin{aligned}&\int_\Omega\nabla_xU^\varepsilon\cdot\nabla_x\check{V}^\varepsilon\,\mathrm{d}x+\int_0^\ell\int_0^{h(\tau)}\partial_\tau\mathsf{u}^\varepsilon\partial_\tau v K_\varepsilon^{-1}\,\mathrm{d}\zeta\,\mathrm{d}\tau+\int_\Omega U^\varepsilon\check{V}^\varepsilon\,\mathrm{d}x+\int_0^\ell\int_0^{h(\tau)}\mathsf{u}^\varepsilon vK_\varepsilon\,\mathrm{d}\zeta\,\mathrm{d}\tau\\&\quad=\int_\Omega\check{F}^\varepsilon\check{V}^\varepsilon\,\mathrm{d}x+\int_0^\ell\int_0^{h(\tau)}fvK_\varepsilon\,\mathrm{d}\zeta\,\mathrm{d}\tau,\end{aligned} \tag{6.9}$$

with $\check{F}^\varepsilon$ and $\check{V}^\varepsilon$ functions defined by (6.1). On account of (6.4) and the fact that K_ε and K_ε^{-1} converge towards 1 in $L^2(\Pi)$ when $\varepsilon\to0$, we take limits in (6.9) and we have:

$$\int_\Omega\nabla_xU^*\cdot\nabla_xV\,\mathrm{d}x+\int_0^\ell\int_0^{h(\tau)}\partial_\tau\mathsf{u}^*\partial_\tau v\,\mathrm{d}\zeta\,\mathrm{d}\tau+\int_\Omega U^*V\,\mathrm{d}x+\int_0^\ell\int_0^{h(\tau)}\mathsf{u}^*v\,\mathrm{d}\zeta\,\mathrm{d}\tau=\int_\Omega FV\,\mathrm{d}x+\int_\Gamma hfv\,\mathrm{d}\tau.$$

Then, since $\mathsf{u}^*=u^*(\tau)$ in Π, (U^*,u^*) satisfies (4.6) and, in consequence, $(U^*,u^*)=A_0(F,f)$.

Finally, we prove $\|B_\varepsilon\mathsf{R}_\varepsilon(F,f)-\mathsf{R}_\varepsilon A_0(F,f)\|_\varepsilon\to0$ as $\varepsilon\to0$. By virtue of (6.6), (6.1), the change to local variables and the fact that $\mathsf{u}^*=u^*(\tau)$ in Π, we can write:

$$\begin{aligned}&\left\|B_\varepsilon\mathsf{R}_\varepsilon(F,f)-\mathsf{R}_\varepsilon A_0(F,f)\right\|^2_\varepsilon\\&\quad=\int_\Omega\check{F}^\varepsilon\left(U^\varepsilon-\check{U}^*_\varepsilon\right)+\int_0^\ell\int_0^{h(\tau)}f\left(\mathsf{u}^\varepsilon-\mathsf{u}^*\right)K_\varepsilon\,\mathrm{d}\zeta\,\mathrm{d}\tau-\int_\Omega\nabla_x\check{U}^*_\varepsilon\cdot\nabla_x\left(U^\varepsilon-\check{U}^*_\varepsilon\right)\\&\qquad-\int_0^\ell\int_0^{h(\tau)}\partial_\tau\mathsf{u}^*\partial_\tau\left(\mathsf{u}^\varepsilon-\mathsf{u}^*\right)K_\varepsilon^{-1}\,\mathrm{d}\zeta\,\mathrm{d}\tau-\int_\Omega\check{U}^*_\varepsilon\left(U^\varepsilon-\check{U}^*_\varepsilon\right)\mathrm{d}x-\int_0^\ell\int_0^{h(\tau)}\mathsf{u}^*\left(\mathsf{u}^\varepsilon-\mathsf{u}^*\right)K_\varepsilon\,\mathrm{d}\zeta\,\mathrm{d}\tau.\end{aligned}$$

Now, we have that all the terms converge towards zero and property (c) is satisfied.

In a similar way to property (c), we can prove that for any sequence $(F^\varepsilon,f^\varepsilon)$ in $\mathcal{V}_\varepsilon$ such that $\|(F^\varepsilon,f^\varepsilon)\|_\varepsilon$ is bounded by a constant independent of ε, we can extract a subsequence, still denoted by ε, verifying $\|B_\varepsilon(F^\varepsilon,f^\varepsilon)-$

$\mathsf{R}_\varepsilon(V_0, v_0)\|_\varepsilon \to 0$ as $\varepsilon \to 0$, for a certain function $(V_0, v_0) \in \mathcal{H}_0$. Thus, B_ε is uniformly compact, and property (d) of Lemma 2.3 holds.

Applying Lemma 2.3, we have that for each fixed k, λ_k^ε converge towards μ_k^0 when $\varepsilon \to 0$. Moreover, for any eigenvalue μ_k^0 of (2.17) with multiplicity $\varkappa_k$ ($\mu_k^0 = \mu_{k+1}^0 = \cdots = \mu_{k+\varkappa_k-1}^0$), and for any eigenfunction (V, w) of (2.17) associated with μ_k^0 with $|||(V, w)||| = 1$, there is a linear combination $(\widetilde{U}^\varepsilon, \tilde{w}^\varepsilon)$ of eigenfunctions associated with $\{\lambda_i^\varepsilon\}_{i=k}^{k+\varkappa_k-1}$ such that (6.2) holds.

As regards the proof of the last statement in the theorem, we consider the sequence $(U_k^\varepsilon, w_k^\varepsilon)$ of eigenfunctions of (2.13), $\|(U_k^\varepsilon, w_k^\varepsilon)\|_\varepsilon = 1$. Let us recall that on account of the smoothness of Γ, the smoothness of the eigenfunctions $\{U^\varepsilon, u^\varepsilon\}$ of (1.2) holds and we have: $U^\varepsilon \in C^\infty(\overline{\Omega})$, $u^\varepsilon \in C^\infty(\bar{\omega}_\varepsilon)$ (cf., for instance, [2] for precise references). Then, $w^\varepsilon|_\Gamma = \varepsilon^{(1-t)/2} u^\varepsilon|_\Gamma \in H^1(\Gamma)$. Taking into account the change to local coordinates ζ, τ and using a classical argument of diagonalization we extract a subsequence (still denoted by ε) such that $(U_k^\varepsilon, \mathsf{w}_k^\varepsilon) \to (V_k^*, \mathsf{w}_k^*)$ in $H^1(\Omega) \times H^1(\Pi)$-weak, as $\varepsilon \to 0$, where by w^ε we denote the eigenfunctions w^ε written in the local coordinates. Besides, $V_k^* \in H_0^1(\Omega)$ and $\mathsf{w}_k^* = w_k^*(\tau)$ in Π. If we assume that this limit $(V_k^*, \mathsf{w}_k^*) \neq 0$, on account of (6.4), (6.5) and the fact that $\lambda_k^\varepsilon \to \mu_k^0$ as $\varepsilon \to 0$ and $\mathsf{w}_k^* = w_k^*(\tau)$ in Π, we identify (V_k^*, w_k^*) with an eigenfunction of (2.17) associated with μ_k^0 by taking limits in (2.13) for $(W, w) = \mathsf{R}_\varepsilon(V, v)$, with any fixed $(V, v) \in H_0^1(\Omega)$. Then, using again the convergence $\lambda_k^\varepsilon \to \mu_k^0$ and (2.13) for $(W, w) = (U_k^\varepsilon, u_k^\varepsilon)$, we have $1 = \|(U_k^\varepsilon, w_k^\varepsilon)\|_\varepsilon^2 \to (\mu_k^0 + 1)[\|U_k^*\|_{L^2(\Omega)}^2 + \|w_k^*\|_{L_h^2(\Gamma)}^2]$ and prove that $(V_k^*, w_k^*) \neq 0$. The fact that the (V_k^*, w_k^*) are orthogonal in $H^1(\Omega) \times H^1(\Gamma)$ for the scalar product (2.6) follows from the orthogonality condition for $(U_k^\varepsilon, w_k^\varepsilon)$. The fact that the set $\{(V_k^*, w_k^*)\}_{k=0}^\infty$ forms a basis of $H_0^1(\Omega) \times H^1(\Gamma)$ is obtained by contradiction since all the eigenvalues of (2.17) have finite multiplicity. Therefore, the theorem is proved. □

Remark 6.1. Justification of asymptotics for the eigenelements of problem $(1.2)_1$, $(1.2)_3$–$(1.2)_5$, (1.3), for other possible values of t and m, with $t < 1$ or $m \neq 0$, can be addressed by combining the technique in [2] when $t = 1$ and $m = 0$ along with that introduced in Sections 5 and 6 of the present paper when $t > 1$ and $m = 0$.

6.1. Comparison of the spectra of (1.2) and (1.1)

The following two theorems provide estimates for the discrepancies of the eigenelements of (2.13) and (2.7) when $\delta_\varepsilon = \varepsilon^{t-1}$, $t > 1$.

Theorem 6.2. *There exist constants $\varepsilon_* > 0$ and $C_* > 0$ such that for any eigenvalue μ_l^ε of problem* (2.7) *with $\delta_\varepsilon = \varepsilon^{t-1}$ the restriction $\varepsilon < \varepsilon_*(1 + \mu_l^\varepsilon)^{-1/2}$ provides at least one eigenvalue λ^ε of* (2.13) *such that*

$$\left|\lambda^\varepsilon - \mu_l^\varepsilon\right| < C_* \varepsilon \left(1 + \mu_l^\varepsilon\right)^{3/2}, \tag{6.10}$$

or equivalently, such that

$$\left|\lambda^\varepsilon - \mu_l^\varepsilon\right| < C_l \varepsilon, \tag{6.11}$$

for $\varepsilon \leqslant \varepsilon_l$ with a certain constant C_l independent of ε.

In addition, for each $(\mathbf{U}_l^\varepsilon, \mathbf{w}_l^\varepsilon)$ eigenfunction associated with μ_l^ε, $\{(\mathbf{U}_l^\varepsilon, \mathbf{w}_l^\varepsilon)\}_{l=1}^\infty$ satisfying the normalization condition (2.18), *there are $(\widetilde{U}^\varepsilon, \tilde{w}^\varepsilon) \in H^1(\Omega) \times H^1(\omega_\varepsilon)$, $\|(\widetilde{U}^\varepsilon, \tilde{w}^\varepsilon\|_\varepsilon = 1$, each $(\widetilde{U}^\varepsilon, \tilde{w}^\varepsilon)$ belonging to the eigenspace associated with all the eigenvalues $\lambda_{p(\varepsilon)}^\varepsilon$ of* (2.13) *in the interval $\lambda_{p(\varepsilon)}^\varepsilon \in [\mu_l^\varepsilon - K\varepsilon^\theta, \mu_l^\varepsilon + K\varepsilon^\theta]$, with a fixed $K > 0$ and $0 < \theta < 1$, $(\widetilde{U}^\varepsilon, \tilde{w}^\varepsilon)$ such that*

$$\left\|\left(\widetilde{U}^\varepsilon, \tilde{w}^\varepsilon\right) - \beta^\varepsilon\left(\mathbf{U}_l^\varepsilon, \check{\mathbf{w}}_l^\varepsilon\right)\right\|_\varepsilon \leqslant C_* \varepsilon^{1-\theta}\left(1 + \mu_l^\varepsilon\right)^{-1/2} \tag{6.12}$$

where $\check{\mathbf{w}}_l^\varepsilon$ is defined by (6.1) *and $\beta^\varepsilon = \|(\mathbf{U}_l^\varepsilon, \check{\mathbf{w}}_l^\varepsilon)\|_\varepsilon^{-1}$.*

Proof. Let μ_l^ε and $(\mathbf{U}_l^\varepsilon, \mathbf{w}_l^\varepsilon)$ be as the theorem states. We first provide some useful estimates for $\mathbf{U}_l^\varepsilon$.

The normalization condition (2.18), (2.5) and (2.7) give us:

$$\left(1+\mu_l^\varepsilon\right)^{1/2}\left\|\mathbf{U}_l^\varepsilon\right\|_{L^2(\Omega)}+\left\|\nabla_x\mathbf{U}_l^\varepsilon\right\|_{L^2(\Omega)}\leqslant C, \tag{6.13}$$

$$\left(1+\mu_l^\varepsilon\right)^{1/2}\left\|\mathbf{U}_l^\varepsilon\right\|_{L^2(\Gamma)}+\left\|\partial_\tau\mathbf{U}_l^\varepsilon\right\|_{L^2(\Gamma)}\leqslant C\varepsilon^{(t-1)/2}, \tag{6.14}$$

where here, and throughout all the proof, C denotes a constant independent of ε and l.

On the other hand, we observe that $\mathbf{U}_l^\varepsilon\in C^\infty(\overline{\Omega})$ verifies:

$$\begin{cases}-\Delta_x\mathbf{U}_l^\varepsilon=\mu_l^\varepsilon\mathbf{U}_l^\varepsilon & \text{in } \Omega,\\ \partial_\tau(h\partial_\tau\mathbf{U}_l^\varepsilon)=\varepsilon^{t-1}\partial_\nu\mathbf{U}_l^\varepsilon-\mu_l^\varepsilon h\mathbf{U}_l^\varepsilon & \text{on } \Gamma,\end{cases} \tag{6.15}$$

and, therefore, we can use the estimates in [1] for the solutions of (6.15) and we have

$$\left\|\mathbf{U}_l^\varepsilon\right\|_{H^3(\Omega)}\leqslant C\left(\left\|\mu_l^\varepsilon\mathbf{U}_l^\varepsilon\right\|_{H^1(\Omega)}+\left\|\varepsilon^{t-1}\partial_\nu\mathbf{U}_l^\varepsilon-\mu_l^\varepsilon h\mathbf{U}_l^\varepsilon\right\|_{H^{1/2}(\Gamma)}+\left\|\mathbf{U}_l^\varepsilon\right\|_{L^2(\Omega)}\right).$$

Then, using the trace inequalities $\|\mathbf{U}_l^\varepsilon\|_{H^{1/2}(\Gamma)}\leqslant C\|\mathbf{U}_l^\varepsilon\|_{H^1(\Omega)}$ and $\|\partial_\nu\mathbf{U}_l^\varepsilon\|_{H^{1/2}(\Gamma)}\leqslant C\|\mathbf{U}_l^\varepsilon\|_{H^2(\Omega)}$, along with estimate (6.13), we obtain:

$$\left\|\mathbf{U}_l^\varepsilon\right\|_{H^3(\Omega)}\leqslant C\left(\mu_l^\varepsilon+\varepsilon^{t-1}\left\|\mathbf{U}_l^\varepsilon\right\|_{H^2(\Omega)}+\mu_l^\varepsilon+1\right)\leqslant C\left(1+\mu_l^\varepsilon\right)+C\varepsilon^{t-1}\left\|\mathbf{U}_l^\varepsilon\right\|_{H^3(\Omega)},$$

and, for ε sufficiently small, we have

$$\left\|\mathbf{U}_l^\varepsilon\right\|_{H^3(\Omega)}\leqslant C\left(1+\mu_l^\varepsilon\right). \tag{6.16}$$

Estimates (6.13), (6.16) and the multiplicative inequality for the second derivatives of smooth functions V,

$$\left\|D_x^2V\right\|_{L^2(\Omega)}^2\leqslant C\|\nabla_xV\|_{H^2(\Omega)}\|\nabla_xV\|_{L^2(\Omega)},$$

give us

$$\left\|\mathbf{U}_l^\varepsilon\right\|_{H^2(\Omega)}\leqslant C\left(1+\mu_l^\varepsilon\right)^{1/2}. \tag{6.17}$$

Now, we consider the Hilbert space $\mathcal{V}_\varepsilon$ defined by (2.11) and the positive, selfadjoint and compact operator B_ε defined by (6.3) with eigenvalues $\{(1+\lambda_k^\varepsilon)^{-1}\}_{k=0}^\infty$. For $\varepsilon\leqslant\varepsilon_0$, we consider the function $(V^\varepsilon,v^\varepsilon)=(\mathbf{U}_l^\varepsilon,\check{\mathbf{w}}_l^\varepsilon)$ where $\check{\mathbf{w}}_l^\varepsilon$ is defined by (6.1). It is clear that $(V^\varepsilon,v^\varepsilon)\in\mathcal{V}_\varepsilon$. In order to apply Lemma 2.1, we prove estimate:

$$\left|\left(B_\varepsilon\left(\widetilde{V}^\varepsilon,\tilde{v}^\varepsilon\right)-\frac{1}{1+\mu_l^\varepsilon}\left(\widetilde{V}^\varepsilon,\tilde{v}^\varepsilon\right),(G,g)\right)_\varepsilon\right|\leqslant C\varepsilon\left(1+\mu_l^\varepsilon\right)^{-1/2}\left\|(G,g)\right\|_\varepsilon\quad\forall(G,g)\in\mathcal{V}_\varepsilon, \tag{6.18}$$

where $(\widetilde{V}^\varepsilon,\tilde{v}^\varepsilon)=\|(V^\varepsilon,v^\varepsilon)\|_\varepsilon^{-1}(V^\varepsilon,v^\varepsilon)$.

Because of the definition of B^ε and the scalar product $(\cdot,\cdot)_\varepsilon$ we can write:

$$\begin{aligned}&\left(1+\mu_l^\varepsilon\right)\left(B_\varepsilon\left(V^\varepsilon,v^\varepsilon\right)-\frac{1}{1+\mu_l^\varepsilon}\left(V^\varepsilon,v^\varepsilon\right),(G,g)\right)_\varepsilon\\&\quad=\mu_l^\varepsilon\int\limits_\Omega V^\varepsilon G\,\mathrm{d}x+\mu_l^\varepsilon\frac{1}{\varepsilon}\int\limits_{\omega_\varepsilon}v^\varepsilon g\,\mathrm{d}x-\int\limits_\Omega\nabla_xV^\varepsilon\cdot\nabla_xG\,\mathrm{d}x-\frac{1}{\varepsilon}\int\limits_{\omega_\varepsilon}\nabla_xv^\varepsilon\cdot\nabla_xg\,\mathrm{d}x.\end{aligned}$$

Using the curvilinear coordinates in ω_ε and taking into account the definition of $(V^\varepsilon,v^\varepsilon)$, $(6.15)_1$ and the fact $(G,g)\in\mathcal{V}_\varepsilon$, we have:

$$\begin{aligned}&\left(1+\mu_l^\varepsilon\right)\left(B_\varepsilon\left(V^\varepsilon,v^\varepsilon\right)-\frac{1}{1+\mu_l^\varepsilon}\left(V^\varepsilon,v^\varepsilon\right),(G,g)\right)_\varepsilon\\&\quad=\varepsilon^{(1-t)/2}\left(-\varepsilon^{(t-1)}\int\limits_\Gamma\partial_\nu\mathbf{U}_l^\varepsilon g\,\mathrm{d}\tau\right.\\&\qquad\left.-\frac{1}{\varepsilon}\int\limits_0^\ell\int\limits_0^{\varepsilon h(\tau)}\left(\partial_\tau\mathbf{U}_l^\varepsilon(0,\tau)\partial_\tau g(\nu,\tau)K(\nu,\tau)^{-1}+\mu_l^\varepsilon\mathbf{U}_l^\varepsilon(0,\tau)g(\nu,\tau)K(\nu,\tau)\right)\mathrm{d}\nu\,\mathrm{d}\tau\right).\end{aligned} \tag{6.19}$$

Let us denote by $\bar{g} \in H^1(\Gamma)$ the mean value of functions g across the band ω_ε, i.e.

$$\bar{g}(\tau) = \frac{1}{\varepsilon h(\tau)} \int_0^{\varepsilon h(\tau)} g(\nu, \tau)\, \mathrm{d}\nu.$$

Multiplying $(6.15)_2$ by $\bar{w}$ and integrating along Γ yields,

$$\varepsilon^{t-1} \int_0^\ell \partial_\nu \mathbf{U}_l^\varepsilon \bar{g}\, \mathrm{d}\tau + \int_0^\ell h \partial_\tau \mathbf{U}_l^\varepsilon \partial_\tau \bar{g}\, \mathrm{d}\tau - \mu_l^\varepsilon \int_0^\ell h \mathbf{U}_l^\varepsilon \bar{g}\, \mathrm{d}\tau = 0. \tag{6.20}$$

Now, we insert the left-hand side of (6.20) multiplied by $\varepsilon^{(1-t)/2}$ into (6.19) and we obtain:

$$\left(1 + \mu_l^\varepsilon\right)\left(B_\varepsilon\left(V^\varepsilon, v^\varepsilon\right) - \frac{1}{1 + \mu_l^\varepsilon}\left(V^\varepsilon, v^\varepsilon\right), (G, g)\right)_\varepsilon = S_1 + S_2 + S_3,$$

where

$$S_1 = -\varepsilon^{(t-1)/2} \int_0^\ell \partial_\nu \mathbf{U}_l^\varepsilon(0, \tau)\left(g(0, \tau) - \bar{g}(\tau)\right) \mathrm{d}\tau,$$

$$S_2 = \varepsilon^{(1-t)/2} \mu_l^\varepsilon \left(\frac{1}{\varepsilon} \int_0^\ell \int_0^{\varepsilon h(\tau)} \mathbf{U}_l^\varepsilon(0, \tau) g(\nu, \tau) K(\nu, \tau)\, \mathrm{d}\nu\, \mathrm{d}\tau - \int_0^\ell h(\tau) \mathbf{U}_l^\varepsilon(0, \tau) \bar{g}(\tau)\, \mathrm{d}\tau\right),$$

$$S_3 = \varepsilon^{(1-t)/2}\left(-\frac{1}{\varepsilon} \int_0^\ell \int_0^{\varepsilon h(\tau)} \partial_\tau \mathbf{U}_l^\varepsilon(0, \tau) \partial_\tau g(\nu, \tau) K(\nu, \tau)^{-1}\, \mathrm{d}\nu\, \mathrm{d}\tau + \int_0^\ell h(\tau) \partial_\tau \mathbf{U}_l^\varepsilon(0, \tau) \partial_\tau \bar{g}(\tau)\, \mathrm{d}\tau\right).$$

Thus, to derive (6.18) we obtain estimates for each term S_i with $i = 1, 2, 3$.

Taking into account the inequality

$$\left| Z(P) - \frac{1}{T} \int_0^T Z(t)\, \mathrm{d}t \right| \leqslant T^{1/2} \|Z'\|_{L^2(0,T)} \quad \text{for } Z \in H^1(0, T),\ P = 0 \text{ or } P = T > 0,$$

the trace inequality $\|\partial_\nu V\|_{L^2(\Gamma)} \leqslant C\|V\|_{H^2(\Omega)}$, estimates (6.17), (6.13), (6.14), (2.21), (2.22) and the formula for the derivative of the mean value,

$$\partial_\tau \bar{g}(\nu, \tau) = -\frac{h'(\tau)}{\varepsilon h(\tau)^2} \int_0^{\varepsilon h(\tau)} g(\nu, \tau)\, \mathrm{d}\nu + \frac{1}{\varepsilon h(\tau)} \int_0^{\varepsilon h(\tau)} \partial_\tau g(\nu, \tau)\, \mathrm{d}\nu + \frac{\varepsilon h'(\tau)}{\varepsilon h(\tau)} g\left(\varepsilon h(\tau), \tau\right),$$

it follows that

$$|S_1| \leqslant C\varepsilon^{(t-1)/2} \left\|\partial_\nu \mathbf{U}_l^\varepsilon\right\|_{L^2(\Gamma)} \left(\int_0^\ell \varepsilon h(\tau) \int_0^{\varepsilon h(\tau)} \left|\partial_\nu g(\nu, \tau)\right|^2 \mathrm{d}\nu\, \mathrm{d}\tau\right)^{1/2} \leqslant C\varepsilon^{t/2}\left(1 + \mu_l^\varepsilon\right)^{1/2} \|\nabla_x g\|_{L^2(\omega_\varepsilon)},$$

$$|S_2| = \mu_l^\varepsilon \frac{\varepsilon^{(1-t)/2}}{\varepsilon} \left| \int_0^\ell \int_0^{\varepsilon h(\tau)} \mathbf{U}_l^\varepsilon(0, \tau) g(\nu, \tau)\left[K(\nu, \tau) - 1\right] \mathrm{d}\nu\, \mathrm{d}\tau \right| \leqslant C\varepsilon^{1/2}\left(\mu_l^\varepsilon\right)^{1/2} \|g\|_{L^2(\omega_\varepsilon)},$$

$$|S_3| \leqslant \varepsilon^{(1-t)/2} \left| \int_0^\ell h'(\tau) \partial_\tau \mathbf{U}_l^\varepsilon(0, \tau)\left(g\left(\varepsilon h(\tau), \tau\right) - \frac{1}{\varepsilon h(\tau)} \int_0^{\varepsilon h(\tau)} g(\nu, \tau)\, \mathrm{d}\nu\right) \mathrm{d}\tau \right|$$

$$+ \frac{\varepsilon^{(1-t)/2}}{\varepsilon} \left| \int_0^\ell \int_0^{\varepsilon h(\tau)} \partial_\tau \mathbf{U}_l^\varepsilon(0, \tau) \partial_\tau g(\nu, \tau)\left[K(\nu, \tau)^{-1} - 1\right] \mathrm{d}\nu\, \mathrm{d}\tau \right| \leqslant C\varepsilon^{1/2} \|\nabla_x g\|_{L^2(\omega_\varepsilon)}.$$

As a result of the above calculations and definition (2.12) of the scalar product $(\cdot,\cdot)_\varepsilon$, we have:

$$\left|\left(B_\varepsilon\left(V^\varepsilon,v^\varepsilon\right)-\left(1+\mu_l^\varepsilon\right)^{-1}\left(V^\varepsilon,v^\varepsilon\right),(G,g)\right)_\varepsilon\right|\leqslant C\varepsilon\left(1+\mu_l^\varepsilon\right)^{-1/2}\left\|(G,g)\right\|_\varepsilon,$$

and hence, estimate (6.18) holds due to the definition of $(\widetilde{V}^\varepsilon,\tilde{v}^\varepsilon)$ and the fact that $\|(\mathbf{U}_l^\varepsilon,\check{\mathbf{w}}_l^\varepsilon)\|_\varepsilon>C_0$ for $\varepsilon\leqslant\varepsilon_0$.

We apply Lemma 2.1 with $A=B_\varepsilon$, $H=\mathcal{V}_\varepsilon$, $\lambda=(1+\mu_l^\varepsilon)^{-1}$, $u=(\widetilde{V}^\varepsilon,\tilde{v}^\varepsilon)$ and $r=C\varepsilon(1+\mu_l^\varepsilon)^{-1/2}$, and we deduce that there exists at least one eigenvalue $\lambda_{p(\varepsilon)}^\varepsilon$ of (2.13) verifying:

$$\left|\left(1+\mu_l^\varepsilon\right)^{-1}-\left(1+\lambda_{p(\varepsilon)}^\varepsilon\right)^{-1}\right|\leqslant C\varepsilon\left(1+\mu_l^\varepsilon\right)^{-1/2}. \tag{6.21}$$

Besides, under the restriction in the statement of the theorem, $\varepsilon\leqslant\varepsilon_*(1+\mu_l^\varepsilon)^{-1/2}$, for $\varepsilon_*=\max(\varepsilon_0,(2C)^{-1})$, we deduce $(1+\lambda_{p(\varepsilon)}^\varepsilon)<\widetilde{C}(1+\mu_l^\varepsilon)$ where the constant $\widetilde{C}$ does not depend on ε, and inequality (6.21) converts into (6.10) with a certain constant C_*. (6.11) is a consequence of (6.10) and the fact that $\{\mu_l^\varepsilon\}_\varepsilon$ is bounded by a constant independent of ε.

In addition, if we take $r^*=\varepsilon^\theta$ with $0<\theta<1$, Lemma 2.1 ensures that there exist $(\widetilde{U}^\varepsilon,\tilde{w}^\varepsilon)$ as the theorem states. Therefore, the theorem is proved. □

Theorem 6.3. *Let μ_k^0 be an eigenvalue of problem* (2.17) *with multiplicity $\varkappa_k$, $\mu_{k-1}^0<\mu_k^0=\cdots=\mu_{k+\varkappa_k-1}^0<\mu_{k+\varkappa_k}^0$. For $l=k,\ldots,k+\varkappa_k-1$, let μ_l^ε be the lth eigenvalue of problem* (2.7) *for $\delta_\varepsilon=\varepsilon^{t-1}$, which converge towards μ_k^0 because of Theorem* 4.1. *Then, there is $\varepsilon_k>0$ such that for $\varepsilon<\varepsilon_k$ the eigenvalues $\lambda^\varepsilon=\lambda_{p(\varepsilon)}^\varepsilon$ of* (2.13) *satisfying* (6.11), *with $l=k,\ldots,k+\varkappa_k-1$, range in the set of eigenvalues provided by Theorem* 6.1 *converging towards μ_k^0 as $\varepsilon\to0$, i.e., λ_j^ε with $j=k,\ldots,k+\varkappa_k-1$.*

Proof. First, let us observe that for fixed l, the eigenvalues in the statement, $\lambda^\varepsilon=\lambda_{p(\varepsilon)}^\varepsilon$ of (2.13) satisfying (6.11), depend on l and also verify:

$$\lambda_{p(\varepsilon)}^\varepsilon\to\mu_l^0=\mu_k^0,\quad \text{as }\varepsilon\to0,\ l=k,\ldots,k+\varkappa_k-1. \tag{6.22}$$

Then, taking into account the results of Theorems 4.1, 6.1 and 6.2, we prove the theorem by contradiction using the technique in Theorem 4.3.

Assuming that the result of the statement does not hold, we consider the case where $p(\varepsilon)$ is not bounded with respect to ε. Thus, there exists a subsequence ε' such that $p(\varepsilon')\to\infty$ as $\varepsilon'\to0$. Consequently, for ε' sufficiently small, $p(\varepsilon')>k+\varkappa_k$ and $\lambda_{p(\varepsilon')}^{\varepsilon'}\geqslant\lambda_{k+\varkappa_k}^{\varepsilon'}$. Since $\mu_l^{\varepsilon'}\to\mu_l^0=\mu_k^0$ and $\lambda_{k+\varkappa_k}^{\varepsilon'}\to\mu_{k+\varkappa_k}^0$ as $\varepsilon'\to0$, from (6.10) it follows that $\lambda_{p(\varepsilon')}^{\varepsilon'}\to\mu_k^0$ and $\mu_k^0\geqslant\mu_{k+\varkappa_k}^0$ which contradicts our assumption. Therefore, $p(\varepsilon)$ is bounded by a constant independent of ε.

Therefore, for sufficiently small ε, $p(\varepsilon)$ ranges in a bounded set of natural numbers. If we consider that there exists a fixed natural j, $j\neq k,k+1,\ldots,k+\varkappa_k-1$, and a subsequence ε' such that $\lambda_{p(\varepsilon')}^{\varepsilon'}=\lambda_j^{\varepsilon'}$, then, $\lambda_{p(\varepsilon')}^{\varepsilon'}=\lambda_j^{\varepsilon'}\to\mu_j^0$ as $\varepsilon'\to0$, which contradicts again (6.22). Therefore, for $\varepsilon<\varepsilon_k$, the eigenvalue $\lambda^\varepsilon=\lambda_{p(\varepsilon)}^\varepsilon$ appearing in Theorem 6.2 coincides with one of the eigenvalues λ_j^ε with $j=k,\ldots,k+\varkappa_k-1$, which concludes the proof. □

Appendix A. The inverse-direct reduction procedure

The aim of this section is to present precise bounds for the discrepancies of the eigenelements of (1.2) and (1.1) with those of the limiting problem (2.15)–(2.16). These bounds are expressed, in an explicit way, in terms of the small parameter ε, the eigenvalue number k and some properties of the limiting spectrum related with the distance to the nearest eigenvalue and its multiplicity. We use the direct and inverse reduction method (cf. [6,2]) to derive these estimates. Results in Lemmas 2.1 and 2.2, along with laborious computations which use the technique developed in [2] along with certain tools and test functions introduced throughout Sections 4 and 6 allow us to obtain the bounds stated in Theorems A.1 and A.2 related to problems (1.1) and (1.2) respectively.

Let μ_k^0 be an eigenvalue of problem (2.17) with multiplicity $\varkappa_k$. Since for $\mu_0^0=0$ the multiplicity is $\varkappa_0=1$, without loss of generality we can assume that, for any fixed $k>1$,

$$\mu_{k-1}^0<\mu_k^0=\cdots=\mu_{k+\varkappa_k-1}^0<\mu_{k+\varkappa_k}^0.$$

For each k, we define the numbers d_k by:

$$d_0 = \frac{\mu_1^0}{1+\mu_1^0},$$
$$d_k = d_{k+1} = \cdots = d_{k+\varkappa_k-1} = \min\left(\frac{\mu_k^0+1}{\mu_{k-1}^0+1} - 1, 1 - \frac{\mu_k^0+1}{\mu_{k+\varkappa_k}^0+1}\right) \quad \text{for } k \neq 0. \tag{A.1}$$

We observe that $d_k(1+\mu_k^0)^{-1}$ measures the distance from $(1+\mu_k^0)^{-1}$ to the nearest eigenvalue $(1+\mu_p^0)^{-1}$ with either $p<k$ or $p>k+\varkappa_k-1$.

Theorem A.1. *There exist constants $\varepsilon_1>0$ and $C_1>0$ such that for any eigenvalue μ_k^0 of problem* (2.17) *with multiplicity $\varkappa_k$, $\mu_{k-1}^0<\mu_k^0=\cdots=\mu_{k+\varkappa_k-1}^0<\mu_{k+\varkappa_k}^0$, the restriction*

$$\delta_\varepsilon^{1/2} < \varepsilon_1 \varkappa_k^{-1}\left(1+\mu_k^0\right)^{-1}(1+1/d_k)^{-1}, \tag{A.2}$$

where d_k are defined by (A.1), *provides that the eigenvalues μ_j^ε of* (2.7) *verify*:

$$\left|\mu_j^\varepsilon - \mu_k^0\right| \leqslant C_1\delta_\varepsilon^{1/2}\left(1+\mu_k^0\right)^{3/2} \quad \text{for } j=k,\ldots,k+\varkappa_k-1.$$

In addition, if $\{(V_j,w_j)\}_{j=k}^{k+\varkappa_k-1}$ are the eigenfunctions of (2.17) *corresponding to μ_k verifying $(((V_j,w_j),(V_i,w_i)))=\delta_{i,j}$ and if $\{(\mathbf{U}_j^\varepsilon,\mathbf{w}_j^\varepsilon)\}_{j=k}^{k+\varkappa_k-1}$ are the eigenfunctions of* (2.7) *$\mu_k^\varepsilon,\ldots,\mu_{k+\varkappa_k-1}^\varepsilon$ such that $(((\mathbf{U}_j^\varepsilon,\mathbf{w}_j^\varepsilon),(\mathbf{U}_i^\varepsilon,\mathbf{w}_i^\varepsilon)))=\delta_{i,j}$, then, under the restriction* (A.2) *for ε and k, there exist coefficients $a_q^{(j)}(\varepsilon)$ and $\beta_q^{(j)}(\varepsilon)$ such that*

$$\left|\!\left|\!\left|\left(\mathbf{U}_q^\varepsilon,\mathbf{w}_q^\varepsilon\right) - \sum_{j=k}^{k+\varkappa_k-1}\beta_q^{(j)}(\varepsilon)(V_j,w_j)\right|\!\right|\!\right| \leqslant C_2\delta_\varepsilon^{1/2}\left(1+\mu_k^0\right)^{1/2}d_k^{-1}\varkappa_k,$$
$$\left|\!\left|\!\left|(V_q,w_q) - \sum_{j=k}^{k+\varkappa_k-1}a_q^{(j)}(\varepsilon)\left(\mathbf{U}_j^\varepsilon,\mathbf{w}_j^\varepsilon\right)\right|\!\right|\!\right| \leqslant C_3\delta_\varepsilon^{1/2}\left(1+\mu_k^0\right)^{1/2}d_k^{-1}\varkappa_k,$$

for $q=k,\ldots,k+\varkappa_k-1$, where C_2, C_3 are constants independent of ε and k.

As regards precise bounds for convergence rates of the eigenelements of (2.13) towards those of the limiting problem (2.17) we state the result in the following theorem (see [2] for the proof in the different case where $t=1$).

Theorem A.2. *There exist constants $\varepsilon_2>0$ and $C_4>0$ such that for any eigenvalue μ_k^0 of problem* (2.17) *with multiplicity $\varkappa_k$, $\mu_{k-1}^0<\mu_k^0=\cdots=\mu_{k+\varkappa_k-1}^0<\mu_{k+\varkappa_k}^0$, the restriction*

$$\varepsilon+\varepsilon^{(t-1)/2} < \varepsilon_2\varkappa_k^{-1}\left(1+\mu_k^0\right)^{-1}(1+1/d_k)^{-1} \tag{A.3}$$

where d_k are defined by (A.1), *provides that the eigenvalues λ_j^ε of* (2.13) *verify*:

$$\left|\lambda_j^\varepsilon-\mu_k^0\right| \leqslant C_4\left(\varepsilon+\varepsilon^{(t-1)/2}\right)\left(1+\mu_k^0\right)^{3/2} \quad \text{for } j=k,\ldots,k+\varkappa_k-1. \tag{A.4}$$

In addition, if $\{(V_j,w_j)\}_{j=k}^{k+\varkappa_k-1}$ are the eigenfunctions of (2.17) *associated with μ_k verifying* (2.18) *and if $\{(U_j^\varepsilon,w_j^\varepsilon)\}_{j=k}^{k+\varkappa_k-1}$ are the eigenfunctions of* (2.13) *corresponding to $\lambda_k^\varepsilon,\ldots,\lambda_{k+\varkappa_k-1}^\varepsilon$ such that $((U_j^\varepsilon,w_j^\varepsilon),(U_i^\varepsilon,w_i^\varepsilon))_\varepsilon=\delta_{i,j}$, then, under the assumption* (A.3) *for ε and k, there exist coefficients $a_q^{(j)}(\varepsilon)$ and $\beta_q^{(j)}(\varepsilon)$ such that*

$$\left\|\left(U_q^\varepsilon,w_q^\varepsilon\right) - \sum_{j=k}^{k+\varkappa_k-1}\beta_q^{(j)}(\varepsilon)(\check{V}_j,\check{w}_j)\right\|_\varepsilon \leqslant C_5\left(\varepsilon+\varepsilon^{(t-1)/2}\right)\left(1+\mu_k^0\right)^{1/2}d_k^{-1}\varkappa_k, \tag{A.5}$$
$$\left\|(\check{V}_q,\check{w}_q) - \sum_{j=k}^{k+\varkappa_k-1}a_q^{(j)}(\varepsilon)\left(U_j^\varepsilon,w_j^\varepsilon\right)\right\|_\varepsilon \leqslant C_6\left(\varepsilon+\varepsilon^{(t-1)/2}\right)\left(1+\mu_k^0\right)^{1/2}d_k^{-1}\varkappa_k,$$

for $q = k, \dots, k + \varkappa_k - 1$, *where* C_5, C_6 *are constants independent of* ε *and* k, *and* $(\check{V}_j, \check{w}_j) = \|(\check{V}_j^\varepsilon, \check{w}_j^\varepsilon)\|_\varepsilon^{-1}(\check{V}_j^\varepsilon, \check{w}_j^\varepsilon)$ *with* $(\check{V}_j^\varepsilon, \check{w}_j^\varepsilon)$ *the functions defined by* (6.1).

Remark A.1. On account of the results in Theorems 6.2, 6.3 and A.2 we can assert that, at least for $1 < t < 3$, the eigenelements of problem (2.7) with $\delta_\varepsilon = \varepsilon^{t-1}$ provide better approximations for the eigenelements of (1.2) than those of problem (2.17) (cf. (A.4), (A.5) and (6.11), (6.12)).

References

[1] S. Agmon, A. Douglis, L. Nirenberg, Estimates near the boundary for solutions of elliptic partial differential equations satisfying general boundary conditions I, Comm. Pure Appl. Math. 12 (1959) 623–727.
[2] D. Gómez, M. Lobo, S.A. Nazarov, E. Pérez, Spectral stiff problems in domains surrounded by thin bands: asymptotic and uniform estimates for eigenvalues, J. Math. Pures Appl. 85 (2006) 598–632.
[3] T. Kato, Perturbation Theory for Linear Operators, Springer-Verlag, Berlin, 1966.
[4] M. Lobo, E. Pérez, Local problems for vibrating systems with concentrated masses: a review, C. R. Mecanique 331 (2003) 303–317.
[5] V.V. Lukyanov, A.I. Nazarov, Solution of the Venttsel problem for the Laplace and the Helmholtz equations with the help of iterated potentials, Zap. Nauchn. Sem. S.-Peterburg. (POMI) 250 (1998) 203–218, English transl.: J. Math. Sci. (New York) 102 (4) (2000) 4265–4274.
[6] S.A. Nazarov, Asymptotic Theory of Thin Plates and Rods, vol. 1: Dimension Reduction and Integral Estimates, Nauchnaya Kniga, Novosibirsk, 2002.
[7] S.A. Nazarov, Two-term asymptotics of solutions of spectral problems with singular perturbations, Mat. Sb. 181 (3) (1991) 291–320, English transl.: Math. USSR Sbornik 69 (2) (1991) 307–340.
[8] S.A. Nazarov, B.A. Plamenevskii, Asymptotics of the spectrum of the Neumann problem in singularly perturbed thin domains, Algebra Analiz. 2 (2) (1990) 85–111, English transl.: Leningr. Math. J. 2 (2) (1991) 287–311.
[9] M. Lobo, S.A. Nazarov, E. Pérez, Eigenoscillations of contrasting non-homogeneous elastic bodies. Asymptotic and uniform estimates for eigenvalues, IMA J. Appl. Math. 70 (2005) 419–458.
[10] O.A. Oleinik, A.S. Shamaev, G.A. Yosifian, Mathematical Problems in Elasticity and Homogenization, North-Holland, London, 1992.
[11] G. Panasenko, E. Pérez, Asymptotic partial decomposition of domain for spectral problems in rod structures, submitted to referee.
[12] J. Sanchez-Hubert, E. Sanchez-Palencia, Vibration and Coupling of Continuous Systems. Asymptotic Methods, Springer-Verlag, Heidelberg, 1989.
[13] A.D. Wentzell, On boundary conditions for multi-dimensional diffusion processes, Theory Probab. Appl. 4 (1959) 164–177.